Theoretical and Applied Mechanics Letters 13 (2023) 100470

Contents lists available at ScienceDirect

Theoretical and Applied Mechanics Letters

journal homepage: www.elsevier.com/locate/taml

Letter

Space-time correlations of passive scalar in Kraichnan model

Ping-Fan Yang[a,b], Liubin Pan[c], Guowei He[a,b,*]

[a] State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China
[b] School of Engineering Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[c] School of Physics and Astronomy, Sun Yat-Sen University, Zhuhai 519082, China

ARTICLE INFO

Keywords:
Space-time correlation
Passive scalar
Kraichnan model

ABSTRACT

We consider the two-point, two-time (space-time) correlation of passive scalar $R(r,\tau)$ in the Kraichnan model under the assumption of homogeneity and isotropy. Using the fine-gird PDF method, we find that $R(r,\tau)$ satisfies a diffusion equation with constant diffusion coefficient determined by velocity variance and molecular diffusion. Its solution can be expressed in terms of the two-point, one time correlation of passive scalar, i.e., $R(r,0)$. Moreover, the decorrelation of $\hat{R}(k,\tau)$, which is the Fourier transform of $R(r,\tau)$, is determined by $\hat{R}(k,0)$ and a diffusion kernal.

Space-time correlation functions are fundamental for understanding the dynamic coupling between the spatial and temporal scales of motion in turbulent flows [1–4]. It and its Fourier counterpart, the wavenumber-frequency spectra, describe the distributions of physical quantities over both space and time scales. It is notoriously hard to study them due to their complexity, and in this short letter we tackle this problem by investigating the space-time correlation of passive scalar for the case of Kraichnan model flow [5], which is a non-trivial model for chaotic flows with spatial correlation. The Kraichnan model is referred to as the passive scalars advected by a white-in-time velocity field. This model is analytically solvable and thus provides valuable implications to turbulent flows, such as anomalous scaling of structural functions for passive scalars. It has been extensively and successfully used in the studies of passive scalar turbulence [6–11]. Recently, Pagani and Canet [10] have used the functional renormalization group method to study the space-time correlations in turbulent flows. In this work, we use a different method, that is, the fine-grid PDF methods [12] to analytically investigate the space-time correlations of passive scalar in the Kraichnan model flow.

The space-time correlation function of passive scalars θ is a correlation between values of θ at two different points and two different times: $R(\boldsymbol{x},\boldsymbol{y};t,s)=\langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{y},s)\rangle$, where θ could be concentrations and temperatures, etc. In a turbulent flow θ evolves according to the advection-diffusion equation

$$\frac{\partial\theta(\boldsymbol{x},t)}{\partial t}+v_i(\boldsymbol{x},t)\frac{\partial}{\partial x_i}\theta(\boldsymbol{x},t)=\kappa\nabla^2\theta(\boldsymbol{x},t)+f, \tag{1}$$

where $\boldsymbol{v}$ is an incompressible turbulent velocity field, κ is the molecular diffusion coefficient, f is the source term representing the injection of the passive scalar into the flow, which is assumed to be white-noise in time. We start with the formal solution of Eq. (1)

$$\theta(\boldsymbol{x},t)=\theta(\boldsymbol{X}_{t,\boldsymbol{x}}(s),s)+\int_s^t\overline{f\left(\boldsymbol{X}_{t,\boldsymbol{x}}(t'),t'\right)}\mathrm{d}t', \tag{2}$$

where $\boldsymbol{X}_{t,\boldsymbol{x}}(t')$ is the position of a particle at time t', which would move under the advection of the turbulent velocity and the Brownian motion of molecular diffusion at $\boldsymbol{x}$ at time t. $\boldsymbol{X}_{t,\boldsymbol{x}}(s)$ satisfies the equation

$$\frac{\mathrm{d}\boldsymbol{X}_{t,\boldsymbol{x}}(t')}{\mathrm{d}t'}=\tilde{\boldsymbol{v}}\left(\boldsymbol{X}_{t,\boldsymbol{x}}(t'),t'\right)=\boldsymbol{v}\left(\boldsymbol{X}_{t,\boldsymbol{x}}(t'),t'\right)+\sqrt{2\kappa}\boldsymbol{\xi}, \tag{3}$$

with initial condition

$$\boldsymbol{X}_{t,\boldsymbol{x}}(t)=\boldsymbol{x}, \tag{4}$$

where ξ is a Wienner process, i.e., $\overline{\xi_i(t)\xi_j(s)}=\delta_{ij}\delta(t-s)$, corresponding to the Brownian motion of molecular diffusion. The overline symbol $\overline{\cdot}$ denotes the ensemble average of the Wiener process ξ for a given realization of velocity field $\boldsymbol{v}$ and forcing f, so when applying the operation $\overline{\cdot}$, the fields $\boldsymbol{v}$ and f are all frozen for each realization of ξ.

We now consider the space-time correlation function at two particular times, t and s. Without loss of generality, we assume $s<t$. Using Eq. (2), we have

$$\langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{y},s)\rangle_{vf}=\left\langle\left[\int_s^t\overline{f(\boldsymbol{X}_{t,\boldsymbol{x}}(t'),t')}dt'+\theta(\boldsymbol{X}_{t,\boldsymbol{x}}(s),s)\right]\theta(\boldsymbol{y},s)\right\rangle_{vf}. \tag{5}$$

The subscript $\langle\,\rangle_{vf}$ denotes the ensemble average of different realizations of velocity and forcing, and we will drop it in the later derivations for simplicity. We will take time s and the scalar field at s as fixed value, and derive the equation for $\langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{y},s)\rangle$ as a function of the time lag due to the evolution of $\theta(\boldsymbol{x},t)$. It is helpful to rewrite Eq. (5) as the fol-

* Corresponding author.
E-mail address: hgw@lnm.imech.ac.cn (G. He).

https://doi.org/10.1016/j.taml.2023.100470
Received 14 August 2023; Received in revised form 29 August 2023; Accepted 29 August 2023
Available online 1 September 2023

lowing form

$$\langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{y},s)\rangle = \left\langle \int \mathrm{d}^3\rho \left[\int_s^t f(\rho,t')\overline{\delta^3(\boldsymbol{\rho}-\boldsymbol{X}_{t,\boldsymbol{x}}(t'))}\mathrm{d}t' + \theta(\rho,s)\overline{\delta^3(\boldsymbol{\rho}-\boldsymbol{X}_{t,\boldsymbol{x}}(s))}\right]\theta(\boldsymbol{y},s)\right\rangle. \tag{6}$$

Now as the source term f is assumed to be white-noise in time, we would have $\langle f(\boldsymbol{\rho},t')\theta(\boldsymbol{y},s)\rangle = 0$ for any $t' > s$, thus in this case the first term on the r.h.s. of Eq. (6) vanishes. Also in Eq. (6), the passive scalar field at time s depends on the velocity field before time s, and the position of particle $\boldsymbol{X}_{t,\boldsymbol{x}}(s)$ depends on the velocity between time s and t. Therefore, for delta-correlated flows like the Kraichnan's model, $\theta(\boldsymbol{\rho},s)$ or $\theta(\boldsymbol{y},s)$ has no correlation with $\boldsymbol{X}_{t,\boldsymbol{x}}(s)$, which yields

$$\langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{y},s)\rangle = \int \mathrm{d}^3\boldsymbol{\rho}\langle\theta(\boldsymbol{\rho},s)\theta(\boldsymbol{y},s)\rangle\left\langle\overline{\delta^3(\boldsymbol{\rho}-\boldsymbol{X}_{t,\boldsymbol{x}}(s))}\right\rangle. \tag{7}$$

Notice that for the case when the passive scalar field is prescribed at time s, Eq. (7) is also satisfied even for velocity fields with finite correlation time, but in general, Eq. (7) is only an assumption when the velocity is not delta-correlated. The term $\left\langle\overline{\delta^3(\boldsymbol{\rho}-\boldsymbol{X}_{t,\boldsymbol{x}}(s))}\right\rangle$ is the probability distribution function (PDF) of particle appears at $\boldsymbol{\rho}$ at time s given that it is at position $\boldsymbol{x}$ at time t, that is, the backward dispersion PDF of single particle in turbulent flows. We denote it $p(\boldsymbol{\rho},s;\boldsymbol{x},t) \equiv \left\langle\overline{\delta^3(\boldsymbol{\rho}-\boldsymbol{X}_{t,\boldsymbol{x}}(s))}\right\rangle$, and for homogeneous isotropic and stationary flows, $p(\boldsymbol{\rho},s;\boldsymbol{x},t) = p(|\boldsymbol{x}-\boldsymbol{\rho}|,t-s) = p(\eta,\tau)$.

Next we try to find the expressions for $p(\eta,\tau)$, using the standard functional method of stochastic calculus we have [7,13]

$$\frac{\partial p(\eta,\tau)}{\partial\tau} = \frac{1}{\eta^2}\frac{\partial}{\partial\eta}(K(\eta,\tau)\eta^2\frac{\partial p(\eta,\tau)}{\partial\eta}), \tag{8}$$

where $K(\eta,\tau) = \int_0^\tau \mathrm{d}\tau' S(\eta,\tau')$, and S is the longitudinal component of the single-point two-time Lagrangian velocity correlation function

$$S(\eta,\tau') = \left\langle\tilde{v}_L(\eta,\tau)\tilde{v}_L(\boldsymbol{X}_{\tau,\eta}(\tau'),\tau')\right\rangle = \left\langle v_L(\eta,\tau)v_L(\boldsymbol{X}_{\tau,\eta}(\tau'),\tau')\right\rangle + 2\kappa\delta(\tau'-\tau). \tag{9}$$

For Kraichnan model with velocity satisfying

$$\left\langle v_i(\boldsymbol{x},t)v_j(\boldsymbol{y},t')\right\rangle = D_{ij}(\boldsymbol{x}-\boldsymbol{y})\delta(t-t'), \tag{10}$$

where $D_{ij}(\boldsymbol{x}-\boldsymbol{y})$ is a diffusivity tensor (see Eqs. (12) and (13) of Ref. [9]), $S = D_{LL}(0)\delta(\tau'-\tau) + 2\kappa\delta(\tau'-\tau)$, thus $K = D_{LL}(0) + 2\kappa$ is a constant and $p(\eta,\tau)$ could be solved as the solution of Eq. (8)

$$p(\eta,\tau) = \frac{1}{8(K\pi\tau)^{3/2}}\mathrm{e}^{-\frac{\eta^2}{4K\tau}}. \tag{11}$$

For homogeneous, isotropic and stationary flows, we have $R(r = |\boldsymbol{x}-\boldsymbol{y}|, \tau = t-s) \equiv \langle\theta(0,0)\theta(r,\tau)\rangle = \langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{y},s)\rangle$. Then from Eqs. (7) and (11) we could obtain

$$\begin{aligned}
R(r,\tau) &= \int \mathrm{d}\boldsymbol{\rho}R(\boldsymbol{r}-\boldsymbol{\rho},0)\frac{1}{8(K\pi\tau)^{3/2}}\mathrm{e}^{-\frac{\rho^2}{4K\tau}} \\
&= \int_0^\pi 2\pi\sin(\theta)\mathrm{d}\theta\int_0^\infty \gamma^2\mathrm{d}\gamma R(\gamma,0)\frac{\mathrm{e}^{-\frac{\gamma^2+r^2+2r\gamma\cos(\theta)}{4K\tau}}}{8(K\pi\tau)^{3/2}} \\
&= \int_0^\infty 2\pi\gamma^2\mathrm{d}\gamma R(\gamma,0)\frac{\mathrm{e}^{-\frac{\gamma^2+r^2}{4K\tau}}}{8(K\pi\tau)^{3/2}}\frac{4K\tau\sinh(\frac{r\gamma}{2K\tau})}{r\gamma} \\
&= \int_0^\infty \gamma\mathrm{d}\gamma R(\gamma,0)\frac{\mathrm{e}^{-\frac{\gamma^2+r^2}{4K\tau}}}{(K\pi\tau)^{1/2}}\frac{\sinh(\frac{r\gamma}{2K\tau})}{r},
\end{aligned} \tag{12}$$

where we denote $\boldsymbol{\gamma} = \boldsymbol{r}-\boldsymbol{\rho}$. Now given the exact expression of $R(\gamma,0)$, we could integrate Eq. (12) and obtain the analytical expressions for $R(r,\tau)$. For example, if $R(\gamma,0) = \mathrm{e}^{-\gamma^2/r_0^2}$, where r_0 is some typical length scale, then we have

$$R(r,\tau) = \frac{\mathrm{e}^{-\frac{r^2}{r_0^2+4K\tau}}}{(1+\frac{4K\tau}{r_0^2})^{3/2}}. \tag{13}$$

On the other hand, from Eqs. (7) and (8) we can deduce that $R(r,\tau)$ satisfies the diffusion equation, and from Eq. (8) we have the integral expression of R

$$R(r,\tau) = \int \mathrm{d}\boldsymbol{\rho}R(\boldsymbol{r}-\boldsymbol{\rho},0)p(\rho,\tau), \tag{14}$$

thus

$$\frac{\partial R(r,\tau)}{\partial\tau} = \int \mathrm{d}\boldsymbol{\rho}R(\boldsymbol{r}-\boldsymbol{\rho},0)\frac{\partial p(\rho,\tau)}{\partial\tau}, \tag{15}$$

and

$$\begin{aligned}
\frac{\partial R(r,\tau)}{\partial\boldsymbol{r}} &= \int \mathrm{d}\boldsymbol{\rho}\frac{\partial}{\partial\boldsymbol{r}}R(\boldsymbol{r}-\boldsymbol{\rho},0)p(\rho,\tau) \\
&= -\int \mathrm{d}\boldsymbol{\rho}\frac{\partial}{\partial\boldsymbol{\rho}}R(\boldsymbol{r}-\boldsymbol{\rho},0)p(\rho,\tau) \\
&= -R(\boldsymbol{r}-\boldsymbol{\rho},0)p(\rho,\tau)|_{boundary} + \int \mathrm{d}\boldsymbol{\rho}R(\boldsymbol{r}-\boldsymbol{\rho},0)\frac{\partial p(\rho,\tau)}{\partial\boldsymbol{\rho}} \\
&= \int \mathrm{d}\boldsymbol{\rho}R(\boldsymbol{r}-\boldsymbol{\rho},0)\frac{\partial p(\rho,\tau)}{\partial\boldsymbol{\rho}},
\end{aligned} \tag{16}$$

from which we have

$$\frac{\partial^2 R(r,\tau)}{\partial r^2} = \int \mathrm{d}\boldsymbol{\rho}R(\boldsymbol{r}-\boldsymbol{\rho},0)\frac{\partial^2 p(\rho,\tau)}{\partial\rho^2}, \tag{17}$$

therefore, from Eq. (8) we know that

$$\frac{\partial R(r,\tau)}{\partial\tau} = \frac{1}{r^2}\frac{\partial}{\partial r}(K(\tau)r^2\frac{\partial R(r,\tau)}{\partial r}). \tag{18}$$

Now if we apply Taylor expansion to $R(r,\tau)$ at point $r = 0$ and $\tau = 0$, we have

$$\begin{aligned}
R(r,\tau) = {} & R(0,0) + \frac{\partial R}{\partial r}(0,0)r + \frac{\partial R}{\partial\tau}(0,0)\tau + \frac{\partial^2 R}{\partial r\partial\tau}(0,0)r\tau + \frac{1}{2}\frac{\partial^2 R}{\partial r^2}(0,0)r^2 \\
& + \frac{1}{2}\frac{\partial^2 R}{\partial\tau^2}(0,0)\tau^2 + \cdots.
\end{aligned} \tag{19}$$

For physical flows R should satisfy $\frac{\partial R}{\partial r}(0,0) = \frac{\partial^2 R}{\partial r\partial\tau}(0,0) = 0$, thus plugging Eq. (19) into Eq. (18) we have $\frac{\partial R}{\partial\tau}(0,0) = 3K(0)\frac{\partial^2 R}{\partial r^2}(0,0)$. From Eq. (9) we can see that if $\kappa \neq 0$, $K(0)$ is nonzero and thus the isolines of $R(r,\tau)$ should be parabolic for small r and τ. When $\kappa = 0$, the value of $K(0)$ will be depended on the velocity field: if the velocity field is delta-correlated, $K(0)$ is nonzero and the isolines should still be parabolic, while if the velocity field is finite-time correlated, $K(0) = 0$ and the isolines of $R(r,\tau)$ are elliptical in this case.

Notice that K is only function of τ, and when K is not a constant, the solution Eq. (11) could still be used after modification. Denoting $\tilde{\tau} = \int_0^\tau K(\tau')\mathrm{d}\tau'$, then $\mathrm{d}\tilde{\tau} = K(\tau)\mathrm{d}\tau$, $\frac{\partial p}{\partial\tilde{\tau}} = \frac{\partial p}{\partial\tau}\frac{1}{K(\tau)} = \frac{1}{\eta^2}\frac{\partial}{\partial\eta}(\eta^2\frac{\partial p(\eta,\tau)}{\partial\eta})$, which is a diffusion equation with constant diffusion coefficient 1. Thus for finite time correlation flows, the PDF solution could be written as

$$p(\eta,\tau) = \frac{1}{8(\pi\tilde{\tau})^{3/2}}\mathrm{e}^{-\frac{\eta^2}{4\tilde{\tau}}}, \qquad \tilde{\tau} = \int_0^\tau K(\tau')\mathrm{d}\tau'. \tag{20}$$

Finally, applying Fourier transform to Eq. (14) yields

$$\hat{R}(k,\tau) = \hat{R}(k,0) \times \frac{\mathrm{e}^{-k^2\tilde{\tau}}}{2\sqrt{2}\pi^{3/2}}, \tag{21}$$

where $\hat{R}(k,\tau) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}\boldsymbol{k}R(r,\tau)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$, and $\frac{\mathrm{e}^{-k^2\tilde{\tau}}}{2\sqrt{2}\pi^{3/2}}$ is the Fourier transform of the PDF solution Eq. (20). Equation (21) is actually applicable to general turbulent flows if we assume the time correlation between passive scalar and velocity field is negligible, which is the assumption we use to simplify Eq. (6) and obtain Eq. (7). In this case, when τ is in the dissipative scale, $K(\tau) \sim \tau$ thus $\tilde{\tau} \sim \tau^2$. While when τ is in the integral scale, $K(\tau) \sim$ Constant thus $\tilde{\tau} \sim \tau$, and for time separation of inertial scale, in general we need to know the behavior of Lagrangian second order structure function. For the special case of Kraichnan model, $K = D_{LL}(0) + 2\kappa$ is a constant and $\tilde{\tau} = K\tau$. Notice that $D_{LL}(0)$ is just v_L^2 (see Eq. (10)), so in this case the diffusion coefficient K is determined by the variance of velocity v_L^2 and the molecular diffusion κ.

In conclusion, we show that for homogeneous and isotropic flows, under the assumptions of Gaussian velocity and weak correlation between velocity and passive scalar, the two-point two-time correlation of passive scalar $R(r,\tau)$ can be expressed in terms of the two-point single time correlation $R(r,0)$ and the PDF of single particle dispersion (see Eqs. (12) and (21)). When applying Taylor expansion to $R(r,\tau)$ at zero point $r=\tau=0$, the first-order derivative $\frac{\partial R}{\partial \tau}(0,0)$ is finite if the flow is delta-correlated. Thus, the isolines of $R(r,\tau)$ are parabolic for short time and small separation in this case. These results could be compared with numerical simulations in future works and provide a cornerstone for future studies of passive scalar in more realistic situations. For example, temperature distribution could be measured in jets and grid turbulence [12,14], and our theoretical approach derived in this work could be readily extended to these cases and compared with experimental results, which would provide valuable insights into the space-time structure of passive scalar in turbulent flows. Other potential applications might include turbulent mixing and combustion.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (NSFC) Basic Science Center Program for "Multiscale Problems in Nonlinear Mechanics" (Grant No. 11988102).

References

[1] G.W. He, J.B. Zhang, Elliptic model for space-time correlations in turbulent shear flows, Phys. Rev. E 73 (2006) 055303.
[2] X. Zhao, G.W. He, Space-time correlations of fluctuating velocities in turbulent shear flows, Phys. Rev. E 79 (2009) 046316.
[3] G.W. He, G.D. Jin, Y. Yang, Space-time correlations and dynamic coupling in turbulent flows, Annu. Rev. Fluid Mech. 49 (2017) 51–70.
[4] T. Wu, G.W. He, Space-time energy spectra in turbulent shear flows, Phys. Rev. Fluids 6 (2021) 100504.
[5] R.H. Kraichnan, Small-scale structure of a scalar field convected by turbulence, Phys. Fluids 11 (1968) 945–953.
[6] M. Chertkov, G. Falkovich, I. Kolokolov, V. Lebedev, Normal and anomalous scaling of the fourth-order correlation function of a randomly advected passive scalar, Phys. Rev. E 52 (1995) 4924–4941.
[7] K. Gawedzki, Theoretical physics at the end of the twentieth century: Lecture notes of the CRM summer school, Banff, Alberta (2002).
[8] A. Celani, M.M. Afonso, A. Mazzino, Point-source scalar turbulence, J. Fluid Mech. 583 (2007) 189–198.
[9] K.R. Sreenivasan, Turbulent mixing: A perspective, Proc. Natl. Acad. Sci. U.S.A. 116 (2018) 18175.
[10] C. Pagani, L. Canet, Spatio-temporal correlation functions in scalar turbulence from functional renormalization group, Phys. Fluids 33 (2021).
[11] A. Gorbunova, C. Pagani, G. Balarac, L. Canet, V. Rossetto, Eulerian spatiotemporal correlations in passive scalar turbulence, Phys. Rev. Fluids 6 (2021) 124606.
[12] S. Pope, Turbulent flows, Cambridge University Press, 2000.
[13] V.I. Klyatskin, Dynamics of stochastic systems, Elsevier, 2005.
[14] Z. Warhaft, Passive scalars in turbulent flows, Annu. Rev. Fluid Mech. 32 (2000) 203–240.