Historical Review on the Roles of Mathematics in the Study of Aerodynamics

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Mathematics and mechanics are twins and were concurrently developing in history. In the times of classic mechanics, mathematicians always were great masters of mechanics. The progresses of aerodynamics with many great mathematicians and applied mathematicians involved further demonstrated the close link between two disciplines in the 20$^{\text{th}}$ century. The present article is primarily focused on the advances of aerodynamics in the period of aeronautical engineering from low to hypersonic speeds. Correspondingly, singular perturbation theories, hodograph method, mixed type and hyperbolic PDE, shock capture scheme in CFD etc. were developing. The persuasive facts became additional paradigm of excellent combination of mathematics and mechanics. Finally, we foresee potential significant directions in future compressible flow study and expect further collaboration of scientists in mathematics and mechanics communities.

Keywords: Perturbation method, Hodograph, hyperbolic and mixed type equation, Shock capture scheme.

The success of the first mankind's powered flight by Brother Wright in 1903 was an epoch-making event, marking the start of aeronautics and aerospace era. However, engineers were immediately confronting a challenging issue of how to scientifically design an aircraft. Since the famous D'Alemberg paradox implies that the drag of a vehicle based on ideal fluid assumption vanishes and people also had little knowledge about the lift of a wing. The difficulty then was that the Navier-Stokes equation for viscous fluids usually lacks analytical solution even for a simple airfoil and was also unable to be numerically solved prior to the advent of advanced computers (Li 1995).

1. Boundary Layer and Wing Theories

Prandtl (1904) at the Department of Applied Mechanics, Gottingen University of Germany experimentally discovered in a water flume that the effects of viscosity are merely restricted to a very thin layer adjacent to the solid wall for large Re flows and then proposed in time the boundary layer theory for drag estimation. Considering great contribution in enhancing understanding of viscous flows and remarkably promoting the progress of aeronautical engineering, mechanics community unanimously regarded Prandtl's BL theory as a milestone of modern mechanics. As for lift, Rayleigh was the earliest to explain Magnus effect by additional circulation around a body, which can be determined by Kutta (1902)–Joukowski (1907) condition, in an incoming flow with $L = \rho U_\infty \Gamma$. Although Lanchester initiated the study of lift for a wing of finite span, Prandtl (1918) was the first to present its mathematical theory. The vortex system consisting of bound vortex at the wing surface and free vortex trailing from wing tip and extending downstream was assumed responsible for downwash and lift generation. Prandtl furthermore found that the wing of elliptic eddy distribution has the minimal induced drag.

As a matter of fact, both BL and wing theories belong to the category of singular perturbation (Van Dyke 1964, Li & Zhou 1998). The boundary layer approach dealing with a DE with a small parameter in the highest derivative term was broadly applied and extended known as Matched Asymptotic Expansion Method, which enormously enriched the contents of applied mathematics.

Since then people witnessed the rising of aviation industry manufacturing varieties of commercial and military aircrafts at the flight speed of a few hundred km/h. Under these circumstances, the research of compressible flows was put on agenda.

We may usually divide compressible flows into the following regimes: (1) subsonic flows mean that the density variation can no longer be neglected; (2) transonic and supersonic flow regime is characteristic of the appearance of shock waves and aerothermodynamic effects in the flow field; (3) hypersonic flow regime should consider aerothermochemistry effects due to internal freedom excitation (Von Karman 1963).

2. Subsonic and Transonic Flows

As you know, the relative variation of density in an isentropic flow is proportional to M^2, where M denotes Mach number. People considered compressibility effects firstly with the help of perturbation theory when a thin or slender vehicle flying at low Mach number is concerned with. Then, the drag and lift of a wing and slender body at small angle of attack can be estimated theoretically.

However, as the speed of an aircraft is growing to 700 $\sim$ 800 km/h when the disturbance is no longer negligibly small, the air flow will obey a nonlinear potential equation like:

$$\left(1-\frac{\phi_x^2}{a^2}\right)\phi_{xx}-2\frac{\phi_x\phi_y}{a^2}\phi_{xy}+\left(1-\frac{\phi_y^2}{a^2}\right)\phi_{yy}=0,$$

which turns out sufficiently difficult to solve. The most effective approach in two dimension cases is the hodograph method by virtue of exchanging the positions of dependent and independent variables. For gas jet problem, Chaplygin (1904) derived a hodograph equation in term of stream function ψ with the module q and argument θ of velocity vector taken as new independent variables:

$$q^2\psi_{qq}+q(1+M^2)\psi_q+(1-M^2)\psi_{\theta\theta}=0,$$

which is evidently linear and then the superposition principle can be applied again. Of course, the difficulty now is how to find a counterpart in the physical plan corresponding to the solution in the velocity plan. Anyway, we can list some of solved problems by this approach (Kuo 1954):

1) Ringleb solution representing a compressible flow turning 180° around a flat plate;

2) Subsonic plane jets;

3) Subsonic flows around an elliptic airfoil with or without rotation;

4) Karman-Tsien formula relating pressure coefficients for compressible and incompressible flows around an airfoil as an effective design tool in aeronautical engineering at that time (Von Karman 1941).

When looking at the solutions above, people found that the continuous mixed subsonic and supersonic flow can coexist with local maximum Mach number $M_{\max}=2.5$ for Ringleb solution and $M_{\max}=1.25$ and 1.22 for subsonic flows $M_\infty=0.6$ and 0.7 around an elliptic airfoil of thickness 0.6 (Kuo 1953). Such kind of mixed subsonic and supersonic flows can be maintained until shock waves appear. Therefore, the study of typical mixed type PDE, namely, the Tricomi equation

$$\phi_{\eta\eta}-\eta\phi_{\xi\xi}=0,$$

was used to solve transonic flows around a wedge or in the nozzle.

In reality, some kind of discontinuity may appear in unsteady compressible flows as well as in the mixed supersonic and subsonic flow field. The physical mechanism for a shock wave to emerge is attributed to the accumulation of compressible disturbance. On the other hand, the mathematical cause of discontinuity is the occurrence of limiting lines when the Jacobi determinate of hodograph transformation vanishes. As a result, Tsien and Guo defined the Mach number for sonic region first to appear as the lower critical Mach number, whereas they called the Mach number for shock wave to occur as the upper critical Mach number. Actual shock waves tend to occur at the Mach number between lower and upper critical Mach numbers due to flow instability when a flow transit from supersonic to subsonic speed.

When there is a shock wave in the flow field, potential assumption wouldn't be justified any longer because isentropic and irrotational conditions break down. People turned to examine quasi-linear hyperbolic equation and its solution. Actually, Riemann problem describing the evolution of an initial step discontinuity in one dimensional air can be regarded as the earliest study on compressible flows. Courant & Friedrichs (1948) made a comprehensive summary on the research of two categories of flows with discontinuity, namely one dimensional unsteady flow and two dimensional transonic or supersonic flows. Their classic book dealt with propagation of rarefied and compressible waves, the formation of shock waves, and reflection of gas dynamic waves from a free surface or solid wall (including regular and Mach reflection), wave-wave interaction etc.

Gu *et al.* (1961, 1962, and 1963) mathematically examined initial or boundary value problems for hyperbolic equation system of 1+1 or 2 dimensions with 3 dependent functions. By reducing to an integral equation, their uniformly convergence and thus the existence of local solution were proved. The results were applied to one dimensional, cylindrically or spherically symmetric gas motions driven by a piston, flood evolution in a river and two dimensional supersonic flows. The proposed successive approximate method may as well serve as an effective tool to find solutions. Friedrich's positive symmetric theory (1958) for mixed type DE was further extended to DE of higher dimension by Gu.

3. Compressible Viscous Flows

As far as skin friction and heat transfer are concerned, we have to give up the assumption of ideal gas and employ compressible boundary layer theory for viscous gases, the complexities of which came from apparent increment in internal energy accompanied by density and viscosity variations in thickened BL (Tsien 1938).

In order to simultaneously solve both momentum and energy equations effectively based on the transformation method, people at first handled compressible flows around a flat plate under the assumption:

$$\mu \infty T, \text{ and } \rho\mu = C$$

for perfect gas throughout the boundary layer. If the Dorodnitsyn–Howarth transformation is applied, we are able to derive an ODE system corresponding to them with similarity solutions as below:

$$f''' + ff'' = 0$$

$$g'' + \mathbf{Pr} f g' + \frac{(\gamma - 1)M_\infty^2}{4}\mathbf{Pr} f'^2 = 0.$$

Obviously, f satisfies the Blasius equation and g as the solution of an inhomogeneous linear equation can be explicitly expressed in terms of f. In this way, the skin friction and wall temperature of compressible boundary layer over an insulated flat plate can be given as:

$$C_f = \frac{0.664}{\sqrt{Re_x}}\sqrt{C}, C = (1 + 0.36(\gamma - 1)M_\infty^2\sqrt{Pr})^{-(1-\omega)}, \omega = 0.7 \sim 0.9$$

$$T_w = T_\infty(1 + \frac{\gamma - 1}{2}M_\infty^2\sqrt{Pr}).$$

That is, the friction coefficient is a bit smaller than the value of incompressible one and the recovery temperature at the wall is smaller than stagnation one at moderate Mach number for air, where Pr number represents the ratio of viscous and heat diffusions. In the same way, we are able to solve the problem at an isothermal plate. It is easy to understand that the recovery temperature for insulated plate or peak temperature for isothermal plate is higher for media with larger Pr number. Since the momentum equation and the energy equation take the similar form, analogy theory may help us to deduce some very useful arguments such as the Reynolds analogy relating expressions of velocity and temperature profiles and coefficients of heat transfer and skin friction: $St = 0.5C_f Pr^{2/3}$ (Schlichting 1950)

In the days of breaking through “sonic barrier”, wind tunnel tests revealed that the chief criminal of transonic flight failure was attributed to the occurrence of shock at the surface of an airfoil. And then, scientists paid attention to the critical influences of a shock on its aerodynamic performances to answer how aerodynamic stall took place. Although there were solutions about regular and Mach reflections of an incident shock, they weren't consistent with experiments. Consequently, the interaction between shock wave and boundary layer became the frontier in aerodynamics at that time.

Further including viscous and heat transfer effects based on previous studies, Kuo (1953) assumed a potential outflow with pressure disturbance and a viscous inner layer with proper velocity profile. The momentum integration method gave overall tendency in pressure variation and separation, while the approach of differential equation provided the details of flow pattern in the boundary layer. An approximate solution in theory for shock-laminar boundary layer interaction was qualitatively satisfactory with following conclusions:

1) The interaction may induce apparent variation in flow pattern: A series of oblique shock appear prior to the main shock; there will be a bump nearby the incident point;

2) Pressure disturbance decays exponentially upstream in the distance of tens of momentum thicknesses. However, overpressure takes place right behind the shock and then gradually drops to the value of regular reflection for inviscid fluids;

3) When shock strength is strong enough, there will be separation ahead of incident point with backward flow adjacent to the wall. How the flow separates also depends on M and Re to a certain extent. Sometimes, reattachment can occur.

4. Shock Wave Capturing Schemes in CFD

As you have seen in the previous paragraphs, the most salient feature of unsteady, transonic and supersonic flows is the appearance of shock waves, the capture of which obviously is a formidably hard task. Since the continuous flow field is separated by an unknown shock, prior to and behind which the Rankine-Hugoniot relations between physical quantities should be satisfied, tedious and time-consuming fitting procedures to locate the shock by try and error were commonly followed in 1950s. As a matter of fact, a realistic shock wave has a thickness of length scale about molecular free path when gas viscosity is taken into account. Based on this concept, Von Neumann and Richtmyer (1957) proposed to introduce a term of artificial viscosity:

$$q = \begin{cases} \rho b^2 \triangle x^2 (\frac{\partial u}{\partial x})^2, & \frac{\partial u}{\partial x} < 0 \\ 0, & \frac{\partial u}{\partial x} \geq 0, \end{cases}$$

where b is an adjustable constant when solving the inviscid Euler equation so that the shock wave can be identified as the position of large gradient and thus automatically captured. The target of this technique is (1) the width of shock should be restricted in one grid scale; (2) shock wave relations should be satisfied; (3) the computation in the continuous region will not be affected.

Since 1960s, people found that this kind of artificial viscosity can be introduced numerically by constructing some kinds of FD scheme with numerical dissipation. As a result, the most challenging task for CFD during recent decades was to work out high performance schemes with adequate numerical dissipation to accurately capture shock without virtual oscillation. The major effective approaches were:

(1) Upwind scheme and its high order counterparts;

(2) Flux Vector Splitting Scheme(FVS) for multidimensional problems accounting for disturbance propagation along characteristic directions;

(3) Godunov type scheme (1959) based on the exact solution of the Riemann problem capable of accurately representing wave evolution;

(4) Total Variation Diminish Scheme (TVD) $TV(v^{n+1}) \leq TV(v^n)$ by Harten (1983), a new concept to eliminate nonphysical oscillation;

(5) Essentially No-Oscillation (ENO) (Harten 1989) and Weighted Essentially No-Oscillation Scheme (WENO) (Liu 1994, Jiang & Shu 1996), $TV(v^{n+1}) \leq TV(v^n) + (h^r)$, an idea for reconstruction of high order numerical flux based on adaptive or weighted stencil selection to avoid loss of accuracy.

Pirrozoli (2011) systematically reviewed foregoing advances so that higher accuracy was achieved by upwind scheme along with filtering and physical conservation in smooth region and virtual oscillation was diminished by hybrid scheme and nonlinear filter with the help of varieties of shock sensors in the vicinity of discontinuities. Shock capture in unsteady compressible laminar and turbulent flows with complicated geometry remains to be most attractive topic in this regard.

5. Concluding Remark

In the 21th century, China has achieved great success in aerospace engineering in manned flight, walking out of capsule and rendezvous/docking. The future plans are the establishment of space station, deep space exploration and new vehicles for transportation between space and continents. New issues for compressible flows such as: unsteady complicated flows with separation, vortices, turbulence and their control; thermal environment and protection of near space vehicles and combustion/chemical flows in the scramjet engine etc. are most challenging. We believe that the collaboration between mathematics and mechanics communities will continue to play indispensable roles in the process ahead.

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Postscript: The review article on mathematical roles in the study of aerodynamics is especially dedicated to the professor C.H. Gu for his remarkable contribution in promoting mathematical research and applications in this area. Expecting China's needs in aerospace engineering, he had no hesitation to switchs his majority from differential geometry to PDE when he was studying in Russia by the end of 1950s. A mechanics class in the Department of Mathematics, Fudan University was formed in 1958 as soon as he came back in China. Except for systematically planning fundamental curriculums and editing mechanics textbooks, he himself gave two most important courses on "High Speed Aerodynamics" and "PDE of Mixed Type" and supervised a few seminars on wing theory, etc., which I as a student earnestly attended. Hundreds of professional students were trained during the 50 years to meet the needs of different industrial sectors for this new specialty. Therefore, Professor Gu no doubt was the founder of Fudan's mechanics. On the other hand, his research in hyperbolic and mixed type differential equations was pioneering and outstanding and consists of a significant part of his scientific achievements in PDE and mathematical physics.