A Fast Mathematical Modeling Method for Aerodynamic-Heating Predictions

C. Luo and Z. Jiang

Abstract Prediction of aerodynamic heating under different flight conditions is a critical and challenging step in developing a new hypersonic vehicle. The prediction model usually involves a large number of variables, and this makes genetic programming converge too slow. This paper presents a fast mathematical modeling method, divide and conquer, for aerodynamic-heating predictions. It can use the separability feature of the target model to decompose a high-dimensional function into many low-dimensional sub-functions. The separability is detected by a special algorithm, bi-correlation test (BiCT), and the sub-functions could be determined by general genetic programming (GP) algorithms one by one. Thus the computational cost will be increased almost linearly with the increase of function dimension. This can help to break the curse of dimensionality and greatly improved the convergence speed to get the underlying target models from a set of sample data.

1 Introduction

When developing a new hypersonic vehicle, the prediction of its aerodynamic heating rates under different flight conditions is a critical and challenging step. For typical configurations such as a flat plate, a ball head, or a circular leading

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edge, a number of empirical formulas are available [1]. But these prediction models are not applicable to a general configuration nor to local regions with complex flow including corners and gaps. In these cases, it is of fundamental importance to construct a mathematical model based on wind-tunnel data in conjunction with theory analysis. However, this is a difficult problem since the underlying model generally involves many variables and most of them are nonlinear with respect to the objective (heat flux). Genetic programming (GP) is usually considered as an appropriate method for the problem since it does not impose a priori assumptions and can optimize function structure and coefficients simultaneously. However, the convergence speed of GP might be too slow for large-scale problems that involve a large number of variables.

Many efforts have been devoted trying to improve the performance of GP [6] in several ways. Some suggest replacing its tree-based coding method, for example, with an integer string (grammar evolution) [9] or a parse matrix (parse-matrix evolution) [7]. These techniques can simplify the coding process but help little on improving the convergence speed. Some suggest confining its search space, e.g., to generalized linear space (Fast Function eXtraction) [8]. These techniques can accelerate its convergence speed, even by orders of magnitude. However, the speed is gained at the sacrifice of losing the generality, that is, the result might be only an approximation of the target function.

Note that many existing models for aerodynamic-heating prediction have a special feature, which is called separability in this paper, and this makes it possible to develop a new algorithm for faster convergence. For example, the heat flux coefficient S_t of a flat plate could be formulated as

$$S_t = 2.274 \ \sin(\theta)\sqrt{\cos(\theta)}/\sqrt{\mathrm{Re}_x}, \tag{1}$$

and the heat flux q_s at the stagnation point of a sphere as

$$q_s = 1.83 \times 10^{-4} v^3 \sqrt{\rho/R}\,(1 - h_w/h_s)\,. \tag{2}$$

In Eq. (1), the two independent variables, θ and Re_x, are both separable. In Eq. (2), the first three variables, v, ρ, and R, are all separable, and the last two variables, h_w and h_s, are not separable, but combined separable. The function in Eq. (2) is considered partially separable in this paper.

The separability is a very useful feature, which will be used in this paper to accelerate the algorithm of functional modeling. The separability of multivariable functions is defined in Sect. 2. Sections 3 and 4 describe the work flow of the proposed divide and conquer technique and a special algorithm, bi-correlation test (BiCT), to determine the separability of a function, and the concluding remarks are drawn in Sect. 5.

2 Definition of Separability

The proposed method in this paper is based on a new concept referred to as partial separability. It has something in common with existing separability definitions such as [3, 4], but is not the same. To make it clear and easy to understand, we begin with some illustrative examples. The functions as follows could all be regarded as partially separable:

$$\begin{aligned} z &= 0.8 + 0.6^* \left(u^2 + \cos(u)\right) + \sin(v + w) * (v - w); \\ z &= 0.8 - 0.6^* \left(u^2 + \cos(u)\right) - \sin(v + w) * (v - w); \\ z &= 0.8 + 0.6^* \left(u^2 + \cos(u)\right) * \sin(v + w) * (v - w); \\ z &= 0.8 - 0.6^* (\sin(v + w) * (v - w)) / \left(u^2 + \cos(u)\right) \end{aligned}$$

where u is separable with respect to z, while v and w are not separable, but their combination (v, w) is separable. A simple example of non-separable function is $f(x) = \sin(x_1 + x_2 + x_3)$.

More precisely, the separability could be defined as follows.

Definition 1 A scalar function with n continuous variables $f(x)$ ($f : \mathbb{R}^n \mapsto \mathbb{R}, x \in \mathbb{R}^n$) is said to be partially separable if and only if it can be rewritten as

$$f(x) = c_0 \overset{m}{\underset{i=1}{\otimes}} \varphi_i (I_i x) \tag{3}$$

where the binary operator $\otimes$ could be plus (+), minus (−), or times (×). The submatrix of the identity matrix $I_i \in \mathbb{R}^{n_i \times n}$ and the set $\{I_1, I_2, \cdots, I_m\}$ is a partition of the identity matrix $I \in \mathbb{R}^{n \times n}$, $\sum_{i=1}^{m} n_i = n$. The subfuction φ_i is a scalar function such that $\varphi_i : \mathbb{R}^{n_i} \mapsto \mathbb{R}$.

Definition 2 A scalar function with n continuous variables $f(x)$ ($f : \mathbb{R}^n \mapsto \mathbb{R}, x \in \mathbb{R}^n$) is said to be completely separable if and only if it can be rewritten as Eq. (3) and $n_i = 1$ for all $i = 1, 2, \cdots, m$.

Note that separability is an independent feature of a function. A separable function is not necessarily linear (see Fig. 1).

3 Divide and Conquer Algorithm

The idea of divide and conquer for functional modeling is using the separability feature of the underling target function to simplify the search process. Therefore, the most important and fundamental step is to determine whether the concerned problem is separable (at least partial separable) or not. This process is based on a special algorithm, bi-correlation test (BiCT), which will be described in the next sec-

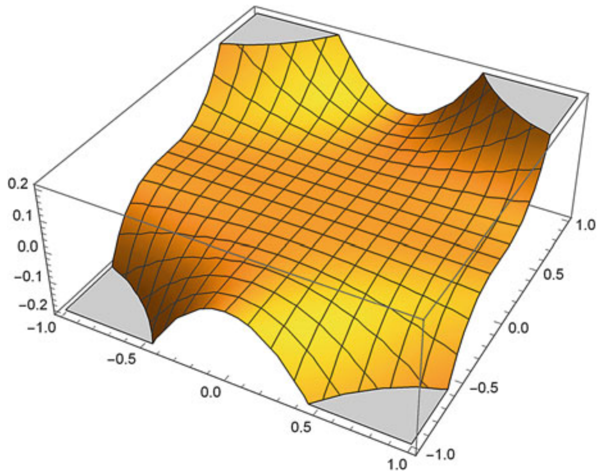

Fig. 1 An example of nonlinear separable functions

tion. Then, based on the separability information, the target function $f(x)$ could be divided into a number of sub-functions with less variables $(\varphi_i(I_i x), i = 1, 2, \cdots, m)$. Next, each sub-function $\varphi_i(I_i x)$ could be determined by some genetic programming algorithm, such as GE [9] and PME [7]. This should be much easier than evolving the target function directly, since the sub-function involves fewer variables and has less complexity. Finally, these sub-functions are properly combined to form the target function.

4 Separability Detection

Studies show that the functional separability defined in the above section could be observed with random sampling and linear correlation techniques. Without the loss of generality, a simple function with three variables $(f(x) = f(x_1, x_2, x_3), x_i \in [a_i, b_i], i = 1, 2, 3)$ is considered to illustrate the implementation of our algorithm. To find out whether the first variable x_1 is separable, two correlation tests are needed.

First, a set of random sampling points in $[a_1, b_1]$ are generated, and then these points are extended to a three-dimensional space with the rest variables (x_2 and x_3) fixed to a point A. We get a vector of function values $f(x_1, A) = \left(f_1^A, f_2^A, \cdots, f_N^A\right) \triangleq f^A$, where N is the number of sampling points. Then these points are extended to a three-dimensional space with fixed x_2 and x_3 to another point B. We get another vector $f(x_1, B) = \left(f_1^B, f_2^B, \cdots, f_N^B\right) \triangleq f^B$. It is obvious that the two vectors f^A and f^B will be linearly correlated if x_1 is separable. However, it is easy to show that this linear correlation could *not* ensure its separability.

Next, it comes to the second correlation test. Another set of random sampling points in $[a_2, b_2] \times [a_3, b_3]$ are generated, and then these points are extended to a three-dimensional space with the rest variable(s) (i.e., x_1) fixed to a point C and get a vector $f(C, x_2, x_3) \triangleq f^C$. Similarly, another vector $f(D, x_2, x_3) \triangleq f^D$ is obtained. Again, the two vectors f^C and f^D need to be linearly correlated to ensure the separability of x_1.

The proposed algorithm is called bi-correlation test (BiCT) since two complementary correlation tests are simultaneously carried out to determine whether a variable or a variable combination is separable. The above process could be easily extended to determine the separability of a function with more variables. The extension process is omitted here. To enhance the stability and efficiency of the algorithm, the distribution of sampling points should be as uniform as possible. Therefore, controlled sampling methods such as Latin hypercube sampling [2] and orthogonal sampling [10] are preferred for sample generation. For the correlation test, Pearson's r method, Spearman's rank-order correlation, and Kendall's τ correlation are all effective.

Take the function $f(x) = 0.8 + 0.6^* \left(x_1^2 + \cos(x_1)\right) * \sin(x_2 + x_3) * (x_2 - x_3), x \in [-3, 3]^3$ as an example, the first sampling set consists of 13 uniformly distributed points in $[-3, 3]$, and the second sampling set consists of 169 uniformly distributed points in $[-3, 3]^2$. The BiCT could be illustrated as in Fig. 2.

For data-driven modeling problems, a surrogate model of black-box type is established as the underlying target function [5] in advance. Then the rest steps are the same as above discussions.

5 Conclusion

A fast mathematical modeling method for aerodynamic-heating predictions, divide and conquer, has been presented. It can use the separability feature of the target model to decompose a high-dimensional function into a series of low-dimensional sub-functions. The separability is detected by a special algorithm, bi-correlation test (BiCT), and the sub-functions could be determined by general genetic programming (GP) algorithms one by one. Thus the computational cost will be increased almost linearly with the increase of function dimension. This can help to break the curse of dimensionality and greatly improved the convergence speed to get the underlying target models from a set of sample data.

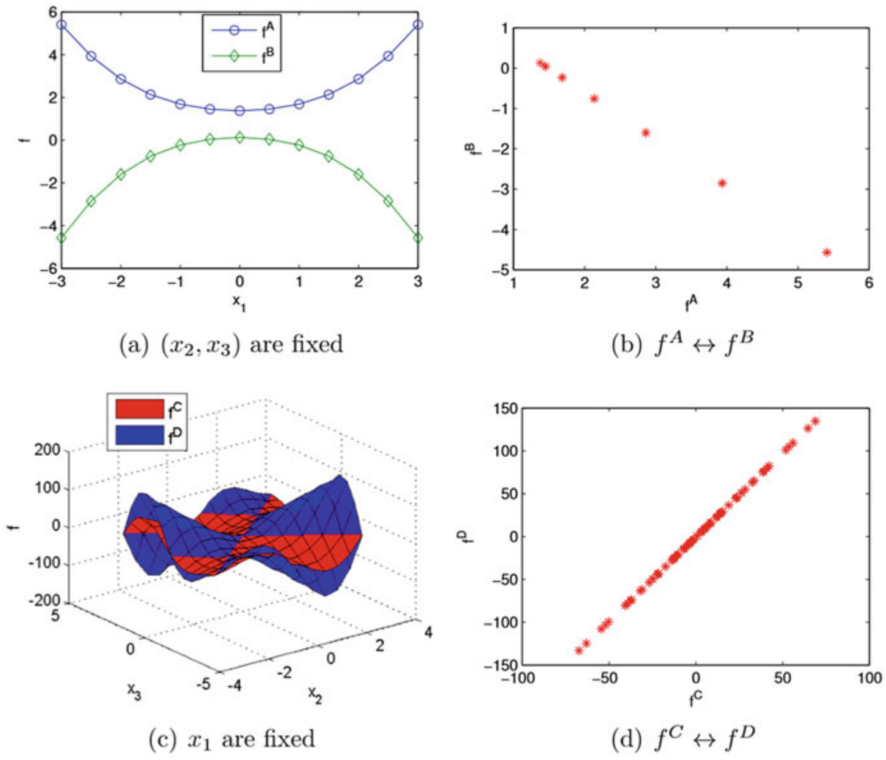

Fig. 2 Demo of separability detection process of BiCT algorithm

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