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Eigenfrequency Shift of Micro-sensor with the Concentrated Mass and Spring

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Abstract **– A new method called finite mode transform method (FMTM) on the eigenfrequency computation of an Euler-Bernoulli beam carrying arbitrary number of concentrated mass and spring is introduced and compared with other existing methods. The FMTM shows good convergence with the increase of the mode number and is a much faster algorithm compared with the analytical method and Galerkin method when the total number of concentrated mass and spring is small. A discussion on the advantages and disadvantages of the microsensor eigenfrequency computation formulation by these three methods (FMTM, analytical and Galerkin) is also presented.**

Index Terms – Eigenfrequency, beam, concentrated mass/spring.

I. INTRODUCTION

Resonator mass sensors operate by providing a frequency shift that is directly proportional to the inertial mass of analyte accreted upon them [1-13]. These sensors are attractive in part because of their high sensitivity and frequency stability, making it possible to detect very small mass changes. The advancements of the fabrication techniques in Micro/Nanoelectromechanical systems (MEMS/NEMS) now push the mass detecting sensitivity limit to the scale of zeptogram (10^{-21} g) [2, 8]. The nanomechanical resonators with the capability of detecting a single *Escherichia coli (E. coli)* bacteria [7, 9] or a single virus [6] have already been fabricated. These nanomechanical resonators are usually beam or plate type of structure [1-13]. So far the molecular dynamics simulation and initial experiments appear to indicate that the continuum mechanics model breaks down only for the structures with the cross section of the order of tens of lattice constants [1]. In this paper we discuss only the continuum mechanics modeling of the nanomechanical resonator of beam structure.

The modeling plays the central role of interpreting the relation of the eigenfrequency shift and the attached mass. The formulation of the problem from the modeling has a significant impact on the computation as shown later in this paper. Nowadays many sensors are bundled/integrated together to form sensor arrays for the calibration and reliability reasons [11, 14, 15]. Choosing an efficient and accurate formulation on the computation of the eigefrequency shift of those nanomechanical resonators can directly result in the performance improvement of the resonator and its array.

In the modeling aspect, we treat the resonator as a continuum system of an Euler-Bernoulli beam. For simplicity reason, some researchers use single degree of freedom (DOF) model to approximate the continuum system of beam structure [1, 3, 5, 6]. The single DOF approximation model is required to find out the system effective stiffness and mass, while the effective stiffness and mass of a continuum system depend on the external loading including its location and shaking frequency [3]. Higher order resonant modes are demonstrated both experimentally and theoretically to have higher sensitivity to the mass change of the resonator [3, 10]. However, finding the effective stiffness and mass for higher modes are very complex and hard [3]. Continuum mechanics modeling, compared with single DOF modeling needs much larger computation effort but avoids the great trouble of finding the effective mass and stiffness. Usually the attached mass is very small compared with the size of resonator structure and the attached mass can thus be modeled as a concentrated mass. For more general application we also add the concentrated translational spring [16] in our following model development. Recent experiment of an *E.Coli* bacteria adsorption on a mass resonator by Ramos *et al.*[9] shows a surprising 24% eigenfrequency increase instead of decrease. Because adding inertia mass can only decrease the system eigenfrequency, Ramos *et al.*[9] argue that the adsorption of the bacteria also increases the system rigidity and this rigidity increase surpasses the increase of inertia mass, which results in the increase of the system eigenfrequency. Incorporating the concentrated translational spring in the model in essence plays the role of increasing the system rigidity.

Three formulations on the eigenfrequency shift of an Euler-Bernoulli beam carrying arbitrary number of concentrated mass and (translational) spring are presented and compared. The three formulations are the analytical method [17-20], Galerkin method [21] and finite mode transform method (FMTM). FMTM is a new method presented in this paper and it adopts the similar approach of Amba-Rao's formulation [22] on the plate carrying a concentrated mass. In Amba-Rao's formulation [22] the plate is with four edges hinged and double sine functions are used to approximate the spatial part of plate deflection for which the name of finite sine transform method (FSTM) is given [23]. Unlike Amba-Rao's FSTM which can only compute the plate carrying a single concentrated mass and with the plate four edges hinged [23], the FMTM here can compute the beam with any kind of boundary conditions and with arbitrary number of

concentrated mass and spring. The advantages, disadvantages and related concerns on the application of these three methods to the eigenfreuqncy shift computation are also presented.

II. Formulation Of The Eigenfrequency Computation Of An Euler-Bernoulli Beam Carrying Arbitrary Number Of Concentrated Mass And Spring

A. Analytical Method

For the analytical method, the effects of concentrated mass and spring on the eigenfrequencies are incorporated in the boundary conditions. The governing equation is still the one for a uniform beam as follows

$$m\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}=0 \tag{1}$$

m is the mass per unit length of the beam; EI is the beam bending stiffness and w is the beam transverse displacement. Here for comparison reason, we give the example of the concentrated mass M_1 at $x=u_1$ case. The beam transverse displacement is divided into two subdomains and assumed to have the following form

$$w(x,t)=Y(x)e^{i\omega t}=\begin{cases}Y_1(x)e^{i\omega t} & x\le u_1\\ Y_2(x)e^{i\omega t} & x\le u_1\end{cases} \tag{2}$$

Here it needs to be emphasized that ω is the system eigenfrequency with the concentrated mass. Y_1, Y_2 have the following solution forms

$$Y_1(x)=C_1\sin(\Lambda x)+C_2\cos(\Lambda x)+C_3\sinh(\Lambda x)+C_4\cosh(\Lambda x)$$
$$Y_2(x)=C_5\sin(\Lambda x)+C_6\cos(\Lambda x)+C_7\sinh(\Lambda x)+C_8\cosh(\Lambda x) \tag{3}$$

Here $\Lambda^4=\frac{m\omega^2}{EI}$. For the eigenfrequency computation, 8 boundary conditions and matching conditions in total are needed. The 4 boundary conditions at the beam ends (x=0 and L) are given as usual. For a concentrated mass M_1 at $x=u_1$, the 4 matching conditions are given as follows [19]

$$Y_1=Y_2,\ \frac{dY_1}{dx}=\frac{dY_2}{dx},$$
$$\frac{d^2Y_1}{dx^2}=\frac{d^2Y_2}{dx^2},\ EI\left(-\frac{d^3Y_1}{dx^3}+\frac{d^3Y_2}{dx^3}\right)=M_1\omega^2Y_1 \tag{4}$$

Together with 4 boundary conditions (depending on whether the beam is hinged, clamped, free or with certain constraints at the ends), an 8×8 matrix is now formed. To have nontrivial solution (i.e. not all C_i (i=1 to 8) are zero, the matrix determinant has to be zero. Therefore, the eigenfrequency ω is found. Adding a concentrated mass or spring generates 4 new matching conditions similar to Eq. (4). So when there is r concentrated masses and s concentrated springs, $Y(x)$ is divided into $r+s+1$ subdomains and finding the eigenfrequencies is a $4(r+s+1)\times 4(r+s+1)$ matrix eigenvalue computation. The above beam model is Euler-Bernoulli beam, which does not consider the beam or concentrated mass rotary inertia influence on the system.

B. Galerkin Method

The Galerkin method incorporates the effects of the concentrated masses and springs directly in the governing equations as the following

$$m\frac{\partial^2 w}{\partial t^2}+EI\frac{\partial^4 w}{\partial x^4}+\sum_{e=1}^{r}M_e\delta(x-u_e)\frac{\partial^2 w}{\partial t^2}+\sum_{e=1}^{s}K_e\delta(x-v_e)w=0 \tag{5}$$

δ is the Dirac delta function. There are r concentrated masses in total and each has the mass of M_e located at u_e ($e=1\ to\ r$); and there are s concentrated springs in total and each has the stiffness of K_e located at v_e ($e=1\ to\ s$), respectively. The following dimensionless quantities are introduced

$$W=\frac{w}{L},\xi=\frac{x}{L},\mu_e=\frac{u_e}{L},\nu_e=\frac{v}{L}$$
$$\tau=\sqrt{\frac{EI}{mL^4}}t,\alpha_e=\frac{M_e}{mL},\beta_e=\frac{K_eL^3}{EI} \tag{6}$$

Now Eq. (5) is nondimensionalized as the following

$$[1+\sum_{e=1}^{r}\alpha_e\delta(\xi-\mu_e)]\frac{\partial^2 W}{\partial\tau^2}+\frac{\partial^4 W}{\partial\xi^4}+\sum_{e=1}^{s}\beta_e\delta(\xi-\nu_e)W=0 \tag{7}$$

Similarly $W(\xi,\tau)$ has the following solution form

$$W(\xi,\tau)=\sum_{j=1}^{N}a_j\phi_j(\xi)e^{i\omega\tau} \tag{8}$$

N is the mode number and a_j is the unknown constants; ϕ_j is the *j-th* mode shape of a uniform beam, which has the following form and orthogonality property [24]:

$$\phi_j(\xi)=A_j\sin(\kappa_j\xi)+B_j\cos(\kappa_j\xi)+C_j\sinh(\kappa_j\xi)+D_j\cosh(\kappa_j\xi)$$
$$\int_0^1\phi_i\phi_j d\xi=\delta_{ij} \tag{9}$$

κ_j, A_j, B_j, C_j and D_j are constants for the *j-th* mode given in reference [24]. δ_{ij} is the Kronecker delta function. Substitute Eq. (8) into Eq. (7) and use the integration property of Dirac delta function and the mode shape orthogonality property, the following equation is obtained

$$MA=0 \tag{11}$$

$A^T=(a_1,a_2,...,a_N)$, M is a $N\times N$ matrix defined as follows

$$M_{ij} = \kappa_j^4 \delta_{ij} + \sum_{e=1}^{s} \beta_e \phi_i(\nu_e)\phi_j(\nu_e) - \omega^2[\delta_{ij} + \sum_{e=1}^{r} \alpha_e \phi_i(\mu_e)\phi_j(\mu_e)] \quad (12)$$

To have the nontrivial solution, the determinant of M has to be zero and the eigenfrequency of ω is thus found.

C. The Finite Mode Transform Method

For finite mode transform method (FMTM), the governing equation is still Eq. (7) and $W(\xi,\tau)$ is assumed the following form

$$W(\xi,\tau) = \psi(\xi)e^{i\omega\tau} \quad (13)$$

Compared with Eq. (8), it is easy to find out that

$$\psi(\xi) = \sum_{j=1}^{N} a_j \phi_j(\xi) \quad (14)$$

Similarly substitute Eq. (13) into Eq. (7), times ϕ_i and integrate, the following equation is obtained after some simple manipulation

$$\int_0^1 \phi_i(\xi)\{\frac{\partial^4 \psi}{\partial \xi^4} - \omega^2\psi + [\sum_{e=1}^{s} \beta_e \delta(\xi - \nu_e) - \omega^2 \sum_{e=1}^{r} \alpha_e \delta(\xi - \mu_e))]\psi\}d\xi = 0 \quad (15)$$

Substitute $\psi(\xi)$ of Eq. (13) into the first two terms of Eq. (15) and the last two terms related to the Dirac function still keep unsubstituted. By using the orthogonality property of the mode shapes and Dirac delta function integration property, a_j is found as follows

$$a_j = \frac{\omega^2 \sum_{e=1}^{r} \alpha_e \phi_j(\mu_e)\psi(\mu_e) - \sum_{e=1}^{s} \beta_e \phi_j(\nu_e)\psi(\mu_e)}{\kappa_j^4 - \omega^2} \quad (16)$$

Now substitute a_j of Eq. (16) into Eq. (14), ψ is now rewritten as

$$\psi(\xi) = \sum_{e=1}^{r} H_e(\xi)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\xi)\psi(\nu_e) \quad (17)$$

$H_e(\xi)$ and $I_e(\xi)$ are defined as follows

$$H_e(\xi) = \frac{\omega^2 \sum_{j=1}^{N} \alpha_e \phi_j(\mu_e)\phi_j(\xi)}{\kappa_j^4 - \omega^2}$$
$$I_e(\xi) = \frac{\sum_{j=1}^{N} \beta_e \phi_j(\nu_e)\phi_j(\xi)}{\kappa_j^4 - \omega^2} \quad (18)$$

Because Eq.(17) is valid for any ξ and we let $\xi = \mu_e$ (e=1 to r) and $\xi = \nu_e$ (e=1 to s), the following *r*+*s* equations are obtained

$$\begin{cases} \psi(\mu_1) = \sum_{e=1}^{r} H_e(\mu_1)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\mu_1)\psi(\nu_e) \\ \psi(\mu_2) = \sum_{e=1}^{r} H_e(\mu_2)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\mu_2)\psi(\nu_e) \\ \cdots\cdots \\ \psi(\mu_r) = \sum_{e=1}^{r} H_e(\mu_r)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\mu_r)\psi(\nu_e) \\ \psi(\nu_1) = \sum_{e=1}^{r} H_e(\nu_1)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\nu_1)\psi(\nu_e) \\ \psi(\nu_2) = \sum_{e=1}^{r} H_e(\nu_2)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\nu_2)\psi(\nu_e) \\ \cdots\cdots \\ \psi(\nu_s) = \sum_{e=1}^{r} H_e(\nu_s)\psi(\mu_e) - \sum_{e=1}^{s} I_e(\nu_s)\psi(\nu_e) \end{cases} \quad (19)$$

The equation above can be rewritten as the following form

$$GV = 0 \quad (20)$$

G is a $(r+s)\times(r+s)$ matrix defined as follows

$$G(\omega) = \begin{pmatrix} H_1(\mu_1)-1 & H_2(\mu_1) & \ldots & H_r(\mu_1) & -I_1(\mu_1) & \ldots & -I_s(\mu_1) \\ H_1(\mu_2) & H_2(\mu_2)-1 & \ldots & H_r(\mu_2) & -I_1(\mu_2) & \ldots & -I_s(\mu_2) \\ \cdots & & & & & & \\ H_1(\mu_r) & H_2(\mu_r) & \ldots & H_r(\mu_r)-1 & -I_1(\mu_r) & \ldots & -I_s(\mu_r) \\ H_1(\nu_1) & H_2(\nu_1) & \ldots & H_r(\nu_1) & -I_1(\nu_1)-1 & \ldots & -I_s(\nu_1) \\ \cdots & & & & & & \\ H_1(\nu_s) & H_2(\nu_s) & \ldots & H_r(\nu_s) & -I_1(\nu_s) & \ldots & -I_s(\nu_s)-1 \end{pmatrix} \quad (21)$$

And $V^T = [\psi(\mu_1), \psi(\mu_2), \ldots, \psi(\mu_r), \psi(\nu_1), \psi(\nu_2), \ldots, \psi(\nu_s)]$

Similarly, for the nontrivial solution, that is not all the masses and springs are at the beam nodes, the determinant of G is required to be zero and the eigenfrequencies are thus found.

III. Computation Examples And Discussions

In order to compare with the results obtained by Low [20], the case of a beam with a single concentrated mass is presented. Here λ_j defined as $\lambda_j = \sqrt{\omega_j}$ is introduced for comparison reasons. In tables I and II the concentrated mass is fixed at $\mu_1 = 0.1$ and the beam is clamped-clamped at the ends. In these two tables the λ_1 values of analytical method are all taken from Low's paper [20].

In Table I, ten modes ($N = 10$) are used in the FMTM computation. Clearly the λ_1 values obtained by the analytical method and FMTM agree with each other relatively well. The accuracy of FMTM is dependent on the mode number. To

demonstrate this we define the error as $\frac{\lambda_1^{FMTM} - \lambda_1^{Analytical}}{\lambda_1^{Analytical}}$ and Fig. 1 plots this error as a function of mode number N when α_1 =20 and 100.

TABLE I
λ_1 VALUES OBTAINED BY ANALYTICAL AND FMTM T

Method/ α_1	0.5	1.0	2.0	10.0	20.0
Analytical	4.708	4.685	4.634	4.123	3.64
FMTM	4.708	4.685	4.634	4.132	3.653

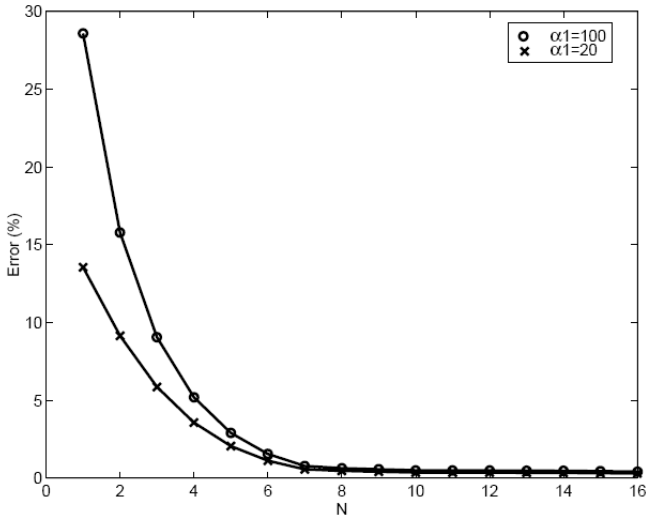

Fig. 1 Convergence study of FMTM as mode number increases.

TABLE II
λ_1 VALUES OBTAINED BY ANALYTICAL AND GALERKIN METHOD WITH DIFFERENT MODE NUMBERS

Method/ α_1	0.01	0.1	1.0	10.0	20.0
Analytical	4.73	4.726	4.685	4.123	3.64
Galerkin (N=1)	4.73	4.726	4.689	4.382	4.133
Galerkin (N=2)	4.73	4.726	4.684	4.149	3.707
Galerkin (N=3)	4.73	4.726	4.668	3.693	3.176

Fig. 1 also plays the role of convergence study. As shown clearly in Fig.1 the error monotonously decreases when the mode number increases. It is worth mentioning that in Table I α_1 starts with 0.5 which is relatively big in micro-sensor application for the detection of very small object attached to it. As mentioned above to have the nontrivial solution requires that not all the concentrated masses and springs locate at the beam nodes and theoretically as far as this condition is satisfied, can the eigenvalues be computed. However, in FMTM when the concentrated mass/spring is small or around the node, the FMTM formulation of Eq. (19) or (20) will experience a $\frac{0}{0}$ type of numerical computation difficulty. The detailed analysis on this is relatively long and not presented here due to the page limitation of this conference paper.

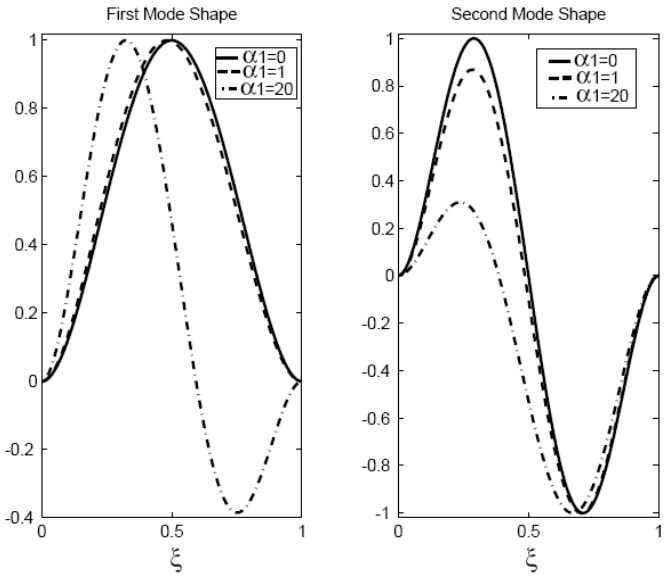

Fig. 2 The 1st and 2nd mode shapes with different α_1 s.

Table II compares the λ_1 values computed by the analytical method and Galerkin method with different mode numbers. Clearly when the concentrated is relatively small, the results obtained by Galerkin method with different mode numbers are all relatively accurate as compared with those obtained by the analytical method. When the concentrated mass becomes large, the eigenvalues obtained from the Galerkin merthod deviates more and more from those of analytical method. This can be explained by the distortion of the mode shapes. As shown in Fig. 2 the first and second mode shapes of the beam with a concentrated mass still at $\mu_1 = 0.1$ and α_1 is taken as 0, 1 and 20, respectively. α_1 =0 is the case of a uniform beam. Clearly the mode shapes of α_1 =1 slightly deviate from those of uniform beam, therefore λ_1 obtained from Galerkin method of N=1 does not significantly differ from that of analytical method. However, the first and second mode shapes of α_1 =20 are dramatically distorted compared with those of uniform beam, especially the first mode which looks more like the second mode of uniform beam. Keep in mind that the Galerkin method is to use the (summation of) mode shapes of uniform beam to approximate the deflection of the beam with the concentrated mass and spring. When the concentrated mass is so large to cause the significant distortion of the mode shapes of uniform beam, to pick one mode (N=1 case) only in the computation is thus not capable of high accuracy. While, it is also noticed that the results obtained by the Galerkin method

of N=2 in general have less error compared with those by Galerkin method of N=3.

IV. CONCLUSIONS

Three different methods on the eigenfrequency computation of an Euler-Bernoulli beam carrying arbitrary number of mass and spring are presented and compared. The analytical method is relatively accurate and capable of all different scenarios' computation but it also has the shortcoming of difficult formulation. When the total number of concentrated mass and spring is $r+s$, the eigenfrequency computation of analytical method is to solve an eigenvalue problem of a $4(r+s+1)\times 4(r+s+1)$ matrix. The formulation of Galerkin method on the eigenfrequency computation is relatively easy and can be done in systematic way. Galerkin method is also capable of different scenarios' computation. The eigenfrequency computation of Galerkin method depends on the mode number and it is an eigenvalue problem of a $N\times N$ matrix (N is the mode number). The shortcoming of the Galerkin method is metioned at the end of the above section that the errors of eigenvalues do not monotonously decrease with the increase of the mode number. The results obtained by the Galerkin method should be carefully examined. Because the Galerkin method uses the mode shapes of uniform beam to approximate and the mode shapes of the beam with concentrated mass and spring can be severely distorted, large number of mode shapes may be needed in the Galerkin method for the accuracy requirement. Although FMTM also uses the mode shapes of uniform beam to approximate, the eigenfrequency computation of FMTM mainly depends on the total number of concentrated mass and spring, which is in essence to solve an eigenvalue problem of a $(r+s)\times(r+s)$ matrix ($r+s$ is the total number of concentrated mass and spring). However, the FMTM method has the numerical difficulty of computing the scenario when the concentrated mass/spring stiffness is small or near the beam node. Therefore, FMTM is not a good computation method for the micro-sensor application of detecting small attached object.

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