Available online at www.sciencedirect.com
ScienceDirect
Procedia Engineering 1 (2009) 135–138
Procedia Engineering
www.elsevier.com/locate/procedia

Mesomechanics 2009

Power law singularity of catastrophic rupture in heterogeneous media

S.W. Hao[a,b,c], M.F. Lu[a], F.J. Ke[a,b], M.F. Xia[a,d], Y.L. Bai[a,*]

[a] *State Key Laboratory of Nonlinear Mechanics (LNM), Institute of Mechanics, CAS, Beijing100190, China*
[b] *Department of Applied Physics, Beihang University, Beijing 100083, China*
[c] *School of Civil Engineering and Mechanics, Yanshan University, Qinhuangdao 066004, China*
[d] *Department of Physics, Peking University, Beijing 100871, China*

Received 19 February 2009; revised 20 April 2009; accepted 21 April 2009

Abstract

Catastrophic rupture in heterogeneous media, such as earthquake in crust, has long been a topic of scientific and societal importance. However, there has been no reliable approach to its prediction. Recently, our experimental study of rocks unveiled a power law singularity ahead of catastrophic rupture, namely the response of the rock specimen R (the ratio of the increments of sample deformation u and controlling crosshead displacement U) can be expressed as $R \sim (1-U/U_F)^{-\beta}$ with $\beta = 0.51 \pm 0.10$ (mean±s.d.) ahead of the catastrophic rupture U_F. Numerical simulations also demonstrate the power law singularity ahead of catastrophic rupture. Theoretical analysis gives the power law singularity with the critical exponent $\beta = 1/2$. In addition, other tests with cement specimens showing gradual failure without catastrophic rupture do not demonstrate such a singularity.

keywords Power law; Singularity; Catastrophic rupture; Heterogeneous media

1. Introduction

Catastrophic rupture, such as rock rupture and earthquake in crust, is a topic of scientific and societal importance[1]. However, there is no reliable approach to prediction owing to its complexity, in particular the uncertain occurrence of catastrophic rupture. It is well known that in both “Nature” and “Science” there have been special debates on whether earthquakes can be predicted or not[2-3]. Since then, what is most closely related to catastrophic rupture becomes a key to the problem. Though a number of options, such as accelerated responses[4-9], have been proposed, the problem on what sort of accelerated criticality can specifically identify the occurrence of catastrophic rupture remains open.

In order to seek such a specific signal, we performed a series of rock tests and examined the variation of the responses near catastrophic rupture. From these tests, a common power law singularity with power law index -0.51±0.10 ahead of the catastrophic rupture was found[10-11]. Further numerical simulations also demonstrate the power law singularity and theoretical analysis gives the power law index -1/2.

In the following, by taking the controlling crosshead displacement U as the governing variable we examine three kinds of responses defined as follows: $R_u=du/dU$, $R_D=dD/dU$ and $R_W=dW/dU$, namely the increments of deformation u, damage fraction D and cumulative energy release W induced by the controlling displacement U respectively.

* Corresponding author. Tel.: +86-10-82543939; fax: +86-10-82543977.
E-mail address: baiyl@lnm.imech.ac.cn.

doi:10.1016/j.proeng.2009.06.031

2. Experimental and Numerical Results

In the experiments, rock samples, 18×20×40 mm^3, were compressed uniaxially. The loading is under the crosshead displacement, U, control. The catastrophic rupture occurs at U_F. Obviously, the controlling displacement consists of the deformation of both loading apparatus and deformed rock sample. The deformation u of the rock sample was measured by an extensometer. Then, the response $R_u=du/dU$ can be obtained each loading step[10-11]. The curves of force P/P_{max} and response $R_u=du/dU$ versus U/U_F are shown in Fig.1, where the subscripts max and F denote maximum load and rupture respectively. All samples show a sharp increase of the response R_u ahead of catastrophic rupture (Fig.1). Now, let us focus on how fast the response $R_u=du/dU$ grows with the controlling displacement U ahead of catastrophic rupture. Fig.2a shows the log-log plots of the response $R_u=du/dU$ versus the reduced displacement $1-U/U_F$ of 3 samples. From the linear dependence in the left part of the log-log plots in this figure, we can see that the increase of the responses ahead of catastrophic rupture can be described by a power law

$$R_u = \frac{du}{dU} \propto \left(1-\frac{U}{U_F}\right)^{-\beta_u} . \tag{1}$$

We fit the power law in the form $R_u=A(1-U/U_F)^{-\beta u}$ to determine the two constants A and β_u. Clearly, this fitting is valid only close to catastrophic rupture. According to the experimental data, a best-fitting of the exponent β_u was found within the last 4%portion ahead of the critical state. Then, we fitted all experimental data of 43 samples of marble and granite to the power law in the range of about 3 orders of magnitude from U/U_F=0.96 to rupture (i.e. $-4 < \log_{10}(1-U/U_F) < -1.4$) and obtained the mean critical exponent $\beta_u=0.51$ with standard error ±0.10. On the other hand, Fig.2b shows the other experiment of cement for which the failure is not catastrophic. Note that there is no singularity at all because no catastrophic rupture occurs, although the response can grow.

By means of multi-scale finite element method[12], we also performed corresponding numerical simulations of the processes from damage to failure in heterogeneous media. Figure 3 shows the numerical results of the variations of two other kinds of response, damage response R_D and energy release response R_W. Interestingly, both of them show the similar power law singularity ahead of catastrophic rupture."

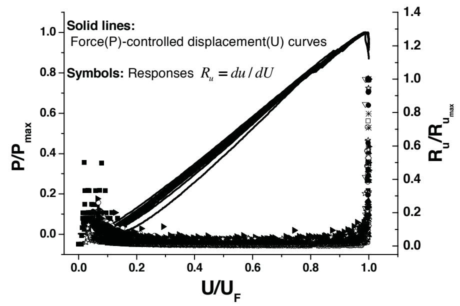

Fig.1 The experimental variations of the normalized load (P/P_{max}) (solid lines) and the normalized responses R_u/R_{umax} for 10 rock samples (symbols) vs controlling displacement (U/U_F). It is clear that the responses grow very fast ahead of catastrophic rupture[11].

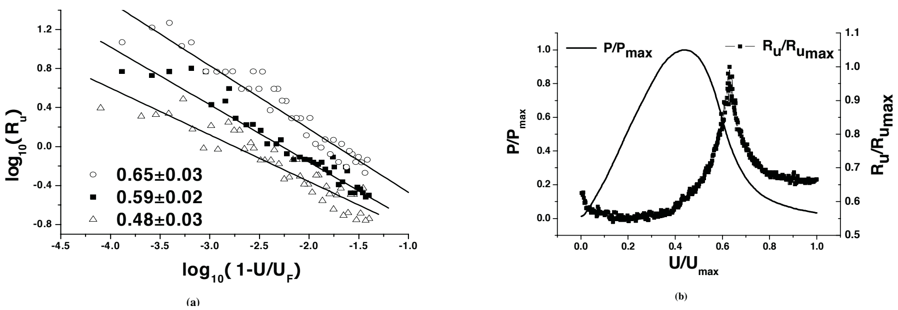

Fig. 2. (a) The log-log plots showing the power law singularity ahead of catastrophic rupture in rock tests (3 samples). Symbols are experimental data and solid lines are their fittings[11]; (b) The variations of the normalized load (P/P_{max}) (solid lines) and the normalized responses R_u/R_{umax} (symbols) vs displacement (U/U_F) for a cement sample. There is no singularity in this case, since its failure is not catastrophic[11].

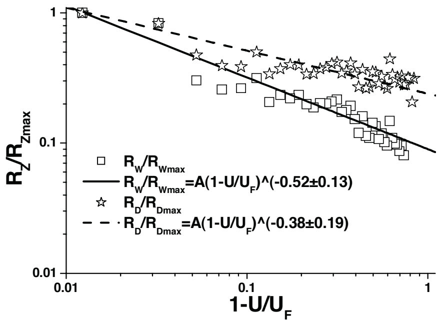

Fig 3. The numerical variations of normalized responses R_D/R_{Dmax} and R_W/R_{Wmax} versus the reduced displacement $1- U/U_F$, showing the power law singularity ahead of catastrophic rupture[11].

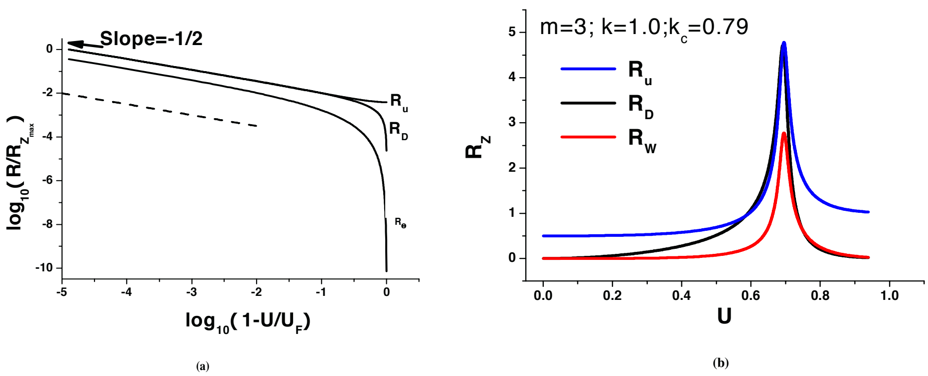

Fig. 4. (a) The analytical log-log plots of normalized responses R_u/R_{umax}, R_D/R_{Dmax} and R_W/R_{Wmax} versus the reduced displacement $1-U/U_F$ for the case of m=2 and k=0.2. The dashed inset with a slope -1/2 is drawn for comparison. Obviously, beyond $log_{10}(1-U/U_F) < -1$ (the left portions of the curves) all responses tend to a power law singularity with power index -1/2,[11]; (b) The analytical variations of normalized responses R_u/R_{umax}, R_D/R_{Dmax} and R_W/R_{Wmax} versus the reduced displacement U/U_F for the case of m=3 and k=1. Since k greater than $k_c=0.79$, there is no catastrophic rupture and no singularity was found for all three responses [11]. It should be noted that the subscript max denotes the cut off values of the responses nearby catastrophic rupture point because of the infinite responses at catastrophic rupture.

3. Analysis based on Mean Field Approximation

Now, we verify the observed power law with the analysis based on mean field approximation[10-11]. It is well known that catastrophic rupture occurs in a sample as soon as the elastic energy released by its environment can fully compensate the rupture dissipation[1]. The system can be considered to consist of two parts in series: an elastic part with stiffness k_e and a damageable sample marked with subscript d and catastrophic rupture occurs only when the stiffness ratio $k=k_e/k_{d0}<k_c$[11], where k_{d0} is the elastic stiffness of the sample. Then, the responses related to the deformation of the damageable sample u can be derived as [11]

$$R_u = \frac{du}{dU} = \frac{k}{k + d\sigma_0/d\varepsilon_d}, \tag{2}$$

where stiffness ratio $k=k_e/k_{d0}=k_e/(E_{d0}A/l_d)$, A is the cross sectional area, l_d is the length and E_{d0} is the initial elastic modulus of the sample. Interestingly, the responses of energy release and damage, R_w and R_D, share the same denominator $k+d\sigma_0/d\varepsilon_d$. So, $\lim_{U\to U_F}(k + d\sigma_0/d\varepsilon_d) = 0$ represents the common singularity of all responses R_Z (Z stands for D, W and u) at catastrophic rupture. After expanding the nominal stress σ_0 and the strain of the damageable sample ε_d and omitting the higher order terms in the vicinity of U_F, one can easily derive[11] $R_Z \propto (1-U/U_F)^{-1/2}$.

After examining a concrete example, namely elastic and statistically brittle (ESB) model with mesoscopic strength ε_c following Weibull distribution $h(\varepsilon_c) = m\varepsilon_c^{m-1}\exp(-\varepsilon_c^{m})$, where m is Weibull modulus, we have calculated all three responses in the whole deformation process[10-11], see Fig. 4a. Clearly, at early stage $-1<log_{10}(1-U/U_F)<0$, all responses do not demonstrate any power law, however, beyond $log_{10}(1-U/U_F) < -1$ all responses tend to a common power law singularity with power law index -1/2. On the other hand, Fig.4b gives a case study showing no catastrophic rupture and no singularity. Now, the comparison of the experimental results in Fig.2 and the theoretical calculations in Fig. 4 can lead to the concluding remark: the power law singularity does specifically appear ahead of catastrophic rupture only. In addition, the *-1/2* power law singularity is independent of stiffness ratio k, the stress-strain relation and the mesoscopic heterogeneity. However, it is worth noting that apart from this universal aspect, the catastrophic rupture shows very sample-specific, namely the catastrophic point U_F is dependent on all factors mentioned above[11]. Additionally, the power law singularity can appear ahead of catastrophic rupture under either crosshead displacement U or load P control, hence the power law singularity can be expressed in a common way $R_Z \propto \left(1-\lambda/\lambda_F\right)^{-1/2}$, where Z and λ stand for cumulative and controlling variables respectively[11].

To summarize this work, a common power law singularity of the responses prior to catastrophic failure is investigated in three different ways: rock experiments, numerical simulations and theoretical analysis. Rock experiments demonstrate a singular power law of the response of deformation $R_u = du/dU \sim (1-U/U_F)^{-\beta u}$ with β_u=0.51±0.10 ahead of catastrophic failure. The numerical simulations also show similar singularity in the responses of damage and energy release. Based on a model with statistical distribution of heterogeneity, mean fields approximation shows that there is always a power law singularity with β=1/2 for all three responses ahead of catastrophic rupture. Furthermore, the power singularity is independent of the details of heterogeneity. On the other hand, both experiments and analysis show that gradual failure does not present such a singularity, though its responses can increase before failure. Therefore, this power law singularity provides not only a good guide to foresee the occurrence of catastrophic rupture, but also to distinguish catastrophic failure from gradual ones.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 90715001，10721202，10802073), National Basic Research Program of China (973 Program 2007CB814803）and CAS Innovation Program.

References

1. Jaeger JC, Cook NGW. *Fundamentals of Rock Mechanics*. 3rd ed. London: Chapman and Hall; 1979.

2. Geller RJ, Jackson DD, Kagan YY, Mulargia F. Earthquakes cannot be predicted. *Science* 1997; **275**:1616-1617.

3. Wyss M. Cannot Earthquakes Be Predicted? *Science* 1997;278:487-490.

4. Weatherley D, Mora P. Accelerating Precursory Activity within a Class of Earthquake Analogue Automata. *Pure Appl. Geophys* 2004; **161**:2005-2019

5. Rundle JB, Turcotte DL, Shcherbakov R, Klein W, Sammis C. Statistical physics approach to understanding the multiscale dynamics of earthquake fault systems. *Rev. Geophys* 2003; **41**, 1019-1048)

6. Ben-Zion Y, Lyakhovsky V. Accelerated seismic release and related aspects of seismicty patterns on earthquake faults. *Pure Appl. Geophys* 2002; **159**:2385-2412.

7. Xia MF, Wei YJ, Ke FJ, Bai YL. Critical Sensitivity and Trans-scale Fluctuations in Catastrophic Rupture. *Pure Appl. Geophys* 2002; **159**:2491-2509.

8. Vere-Jones D, Robinson R, Yang W. Remarks on the accelerated moment release model - Problems of model formulation, simulation and estimation. *Geophys. J. Int* 2001; **144**:517-531.

9. Yin XC, Chen XZ, Song ZP, Yin C. A new approach to earthquake prediction: The load/unload response ratio (LURR) theory. *Pure Appl. Geophys* 1995; **145**:701-715.

10. Hao SW, Wang HY, Xia MF, Ke FJ, Bai YL.Relationship between strain localization and catastrophic rupture. *Theor. Appl. Fract. Mech* 2007; **48**:41-49.

11. Hao SW, Rong F, Lu MF,Wang HY, Xia MF, Ke FJ, Bai YL. A power law singularity as a specific warning of catastrophic rupture in heterogeneous media. (submitted)

12. Rong F, Xia MF, Ke FJ, Bai YL. Adaptive Mesh Refinement FEM for Damage Evolution of Heterogeneous Brittle Media. *Modelling and Simulation in Materials Science and Engineering* 2005; **13**:771-782.