ACTA MECHANICA SINICA, Vol.10, No.1, February 1994 ISSN 0567-7718
Science Press, Beijing, China
Allerton Press, INC., New York, U.S.A.

# A NEW METHOD FOR THE PERIODIC SOLUTION OF STRONGLY NONLINEAR SYSTEMS*

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**ABSTRACT**: A new method for the periodic solution of strongly nonlinear system is given. By using this method, the existance and stability of the periodic solution can be decided, and the approximate expression of the periodic solution can also be found.

**KEY WORDS**: periodic solution, strongly nonlinear systems

## I. INTRODUCTION

The periodic solution of strongly nonlinear system, due to its importance both in theory and application, has become one of the main topics in nonlinear vibration and many valuable results have been obtained. In this paper, a new method for the periodic solution of strongly nonlinear system is given. By using this method, not only the existence and stability of the periodic solution can be decided, but also the approximate expression of the periodic solution can be found. Though this method contains some idea of averaging, it is quite different from the traditional method of averaging. It can be seen in an example that the problem which can not be solved by traditional method of averaging can be solved by this method.

## II. THE METHOD

Consider the strongly nonlinear system

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \tag{2.1}$$

and suppose its solution is of the form

$$x = a\cos(\theta + \theta_0) + b(a) \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = A(a) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = B(a, \theta) \tag{2.2}$$

Received 2 July 1992
* The project supported by the National Natural Science Foundation of China

where $b$ is the deviation of vibration centre from the origin $O$, $B(a,\theta)$ is a periodic function of $\theta$ with period $2\pi$. Expanding $B(a,\theta)$ into Fourier series and considering only harmonics up to $k$-th order we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = B(a,\theta) = \alpha_0(a) + \sum_{n=1}^{k}(\alpha_n(a)\cos n\theta + \beta_n(a)\sin n\theta) \tag{2.3}$$

From (2.2) it follows that

$$\begin{aligned}\dot{x} &= (A\cos\theta_0 - aB\sin\theta_0)\cos\theta - (A\sin\theta_0 + aB\cos\theta_0)\sin\theta + \frac{\mathrm{d}b}{\mathrm{d}a}A\\ \ddot{x} &= [(A\frac{\mathrm{d}A}{\mathrm{d}a} - aB^2)\cos\theta_0 - (aA\frac{\partial B}{\partial a} + aB\frac{\partial B}{\partial\theta} + 2AB)\sin\theta_0]\cos\theta\\ &-[(aA\frac{\partial B}{\partial a} + aB\frac{\partial B}{\partial\theta} + 2AB)\cos\theta_0 + (A\frac{\mathrm{d}A}{\mathrm{d}a} - aB^2)\sin\theta_0]\sin\theta\\ &+A(\frac{\mathrm{d}A}{\mathrm{d}a}\frac{\mathrm{d}b}{\mathrm{d}a} + A\frac{\mathrm{d}^2b}{\mathrm{d}^2a})\end{aligned} \tag{2.4}$$

and from (2.3) we get

$$\begin{aligned}\frac{\partial B}{\partial a} &= \frac{\mathrm{d}\alpha_0}{\mathrm{d}a} + \sum_{n=1}^{k}(\frac{\mathrm{d}\alpha_n}{\mathrm{d}a}\cos n\theta + \frac{\mathrm{d}\beta_n}{\mathrm{d}a}\sin n\theta)\\ \frac{\partial B}{\partial\theta} &= \sum_{n=1}^{k}(-n\alpha_n\sin n\theta + n\beta_n\cos n\theta)\end{aligned} \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1), multiplying it by $\mathrm{d}\theta$, $\cos n\theta\mathrm{d}\theta$, $\sin n\theta\mathrm{d}\theta$ ($n = 1,2,...,k+1$) respectively and then intergrating from $\theta = 0$ to $\theta = 2\pi$, the equations for determining $b, A, \alpha_0, \alpha_n, \beta_n$ ($n = 1,2,...,k$) can be obtained

$$\begin{aligned}&f_0(a,b,A,\alpha_0,\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_k) = 0\\ &f_m(a,b,A,\alpha_0,\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_k) = 0 \qquad (m = 1,2,\cdots,2k+2)\end{aligned} \tag{2.6}$$

After $A(a)$ has been solved from (2.6), the periodic solution of (2.1) can be obtained by putting $A(a)$=0. Suppose $a = a^*$ is a real root of $A(a) = 0$. Substituting $a^*$ into (2.2) and (2.3) the expression of periodic solution is thus obtained.

$$\begin{aligned}x &= a^*\cos(\theta + \theta_0) + b(a^*)\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \alpha_0(a^*) + \sum_{n=1}^{k}(\alpha_n(a^*)\cos n\theta + \beta_n(a^*)\sin n\theta)\end{aligned} \tag{2.7}$$

The harmonic components of the periodic solution can be determined as follows. Differentiating the 1st equation of (2.7) with respect to $t$ and noticing the 2nd equation, we have

$$\begin{aligned}\dot{x} &= (-a^*\cos\theta_0\sin\theta - a^*\sin\theta_0\cos\theta)[\alpha_0(a^*) + \sum_{n=1}^{k}(\alpha_n(a^*)\cos n\theta + \beta_n(a^*)\sin n\theta)]\\ &= -\frac{a^*}{2}\{(\alpha_1(a^*)\sin\theta_0 + \beta_1(a^*)\cos\theta_0) - \alpha_0(a^*)\sin\theta_0\cos\theta - \alpha_0(a^*)\cos\theta_0\sin\theta\\ &+(\alpha_1(a^*)\sin\theta_0 - \beta_1(a^*)\cos\theta_0)\cos 2\theta + (\alpha_1(a^*)\cos\theta_0 + \beta_1(a^*)\sin\theta_0)\sin 2\theta\end{aligned}$$

$$+\sum_{n=2}^{k}[(\alpha_n(a^*)\sin\theta_0-\beta_n(a^*)\cos\theta_0)\cos(n+1)\theta+(\alpha_n(a^*)\sin\theta_0$$
$$+\beta_n(a^*)\cos\theta_0)\cos(n-1)\theta+(\alpha_n(a^*)\cos\theta_0+\beta_n(a^*)\sin\theta_0)$$
$$\sin(n+1)\theta+(-\alpha_n(a^*)\cos\theta_0+\beta_n(a^*)\sin\theta_0)\sin(n+1)\theta]\} \tag{2.8}$$

Since $x$ is a periodic solution, $x$ should not contain the constant term, thus from (2.8), we can get

$$\alpha_1(a^*)\sin\theta_0+\beta_1(a^*)\cos\theta_0=0 \tag{2.9}$$

by which $\theta_0$ in (2.2) can be obtained

$$\sin\theta_0/\cos\theta_0=-\beta_1(a^*)/\alpha_1(a^*) \qquad \theta_0=-\mathrm{tg}^{-1}\beta_1(a^*)/\alpha_1(a^*) \tag{2.10}$$

Now we take the mean value of $\mathrm{d}\theta/\mathrm{d}t$ over a period $2\pi$ as its approximate values, i.e. take $\mathrm{d}\theta/\mathrm{d}t\approx\alpha_0(a^*)$, hence $\theta=\alpha_0(a^*)t$. Substituting it into (2.8), then integrating (2.8) and supplementing the constant $b(a^*)$, we get

$$x=-\frac{a^*}{2}\Big\{-\sin\theta_0\sin\alpha_0(a^*)t+\cos\theta_0\cos\alpha_0(a^*)t$$
$$+\frac{\alpha_1(a^*)\sin\theta_0-\beta_1(a^*)\cos\theta_0}{2\alpha_0(a^*)}\sin 2\alpha_0(a^*)t$$
$$-\frac{\alpha_1(a^*)\cos\theta_0+\beta_1(a^*)\sin\theta_0}{2\alpha_0(a^*)}\cos 2\alpha_0(a^*)t$$
$$+\sum_{n=2}^{k}\Big[\frac{\alpha_n(a^*)\sin\theta_0-\beta_n(a^*)\cos\theta_0}{(n+1)\alpha_0(a^*)}\sin(n+1)\alpha_0(a^*)t$$
$$+\frac{\alpha_n(a^*)\sin\theta_0+\beta_n(a^*)\cos\theta_0}{(n-1)\alpha_0(a^*)}\sin(n-1)\alpha_0(a^*)t$$
$$-\frac{\alpha_n(a^*)\cos\theta_0+\beta_n(a^*)\sin\theta_0}{(n+1)\alpha_0(a^*)}\sin(n+1)\alpha_0(a^*)t$$
$$-\frac{-\alpha_n(a^*)\cos\theta_0+\beta_n(a^*)\sin\theta_0}{(n-1)\alpha_0(a^*)}\cos(n-1)\alpha_0(a^*)t\Big]$$
$$-\frac{\alpha_1(a^*)\cos\theta_0+\beta_1(a^*)\sin\theta_0}{2\alpha_0(a^*)}$$
$$-\sum_{n=1}^{k}\Big[\frac{\alpha_n(a^*)\cos\theta_0+\beta_n(a^*)\sin\theta_0}{(n+1)\alpha_0(a^*)}$$
$$+\frac{-\alpha_n(a^*)\cos\theta_0+\beta_n(a^*)\sin\theta_0}{(n-1)\alpha_0(a^*)}\Big]\Big\}+b(a^*) \tag{2.11}$$

As for the stability of this periodic solution it can be decided by the sign of $\mathrm{d}A(a^*)/\mathrm{d}a$. $\mathrm{d}A(a^*)/\mathrm{d}a<0$ means stable, otherwise unstable.

Generally speaking, it is not easy to solve $A(a)$ from (2.6). If we study only the periodic solution, not the transient process, it is unnecessary to obtain the expression of $A(a)$. Because the periodic solution means that $A(a)=0$, therefore in order to find the periodic solution, we can proceed as follows. In (2.6) put $A=0$, then from these $2k+3$ equations we can solve the $2k+3$ unknowns $a,b,\alpha_0,\cdots,\alpha_k,\beta_1,\cdots,\beta_k$. Suppose these solutions are $\bar{a},\bar{b},\overline{\alpha}_1,\cdots,\overline{\alpha}_k,\overline{\beta}_1,...,\overline{\beta}_k$. Because for these solutions we have $A=0$, then they are just the quantities in (2.7), i.e.

$$\begin{aligned}&\bar{a} = a^* \qquad \bar{b} = b(a^*) \qquad \bar{\alpha}_0 = \alpha_0(a^*)\\ &\bar{\alpha}_1 = \alpha_1(a^*)\cdots\bar{\alpha}_k = \alpha_k(a^*)\\ &\bar{\beta}_1 = \beta_1(a^*)\cdots\bar{\beta}_k = \beta_k(a^*)\end{aligned}$$

Now we go on to study the stability. First expand $a, b, A, \alpha_0, \alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k$ into power series at $a = a^*$

$$\begin{aligned}&a = a^* + (a - a^*)\\ &b(a) = b(a^*) + \frac{\mathrm{d}b(a^*)}{\mathrm{d}a}(a - a^*) + \cdots\\ &A(a) = A(a^*) + \frac{\mathrm{d}A(a^*)}{\mathrm{d}a}(a - a^*) + \cdots = \frac{\mathrm{d}A(a^*)}{\mathrm{d}a}(a - a^*)\\ &\alpha_0(a) = \alpha_0(a^*) + \frac{\mathrm{d}\alpha_0(a^*)}{\mathrm{d}a}(a - a^*) + \cdots\\ &\alpha_n(a) = \alpha_n(a^*) + \frac{\mathrm{d}\alpha_n(a^*)}{\mathrm{d}a}(a - a^*) + \cdots \qquad n = 1, 2, \cdots, k\\ &\beta_n(a) = \beta_n(a^*) + \frac{\mathrm{d}\beta_n(a^*)}{\mathrm{d}a}(a - a^*) + \cdots\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.6), neglecting all the terms $(a-a^*), 1 \geq 2$, noticing that $a^*$, $b(a^*)$, $\alpha_0(a^*)$, ... , $\alpha_n$ $(a^*)$, $\beta_1(a^*)$ , ... , $\beta_n$ $(a^*)$ satisfy (2.6) and eliminating the common factor $(a - a^*)$, then we get

$$\begin{aligned}&f_0(\frac{\mathrm{d}b(a^*)}{\mathrm{d}a}, \frac{\mathrm{d}A(a^*)}{\mathrm{d}a}, \frac{\mathrm{d}\alpha_0(a^*)}{\mathrm{d}a}, \frac{\mathrm{d}\alpha_1(a^*)}{\mathrm{d}a}, \cdots, \frac{\mathrm{d}\alpha_k(a^*)}{\mathrm{d}a},\\ &\qquad \frac{\mathrm{d}\beta_1(a^*)}{\mathrm{d}a}, \cdots, \frac{\mathrm{d}\beta_k(a^*)}{\mathrm{d}a}) = 0\\ &f_m(\frac{\mathrm{d}b(a^*)}{\mathrm{d}a}, \frac{\mathrm{d}A(a^*)}{\mathrm{d}a}, \frac{\mathrm{d}\alpha_0(a^*)}{\mathrm{d}a}, \frac{\mathrm{d}\alpha_1(a^*)}{\mathrm{d}a}, \cdots \frac{\mathrm{d}\alpha_k(a^*)}{\mathrm{d}a},\\ &\qquad \frac{\mathrm{d}\beta_1(a^*)}{\mathrm{d}a}, \cdots, \frac{\mathrm{d}\beta_k(a^*)}{\mathrm{d}a}) = 0 \qquad m = 1, 2, \cdots, 2k + 2\end{aligned} \tag{2.13}$$

From the last $2k+2$ equations of (2.13), we can solve $\mathrm{d}b/\mathrm{d}a$, $\mathrm{d}\alpha_0/\mathrm{d}a$, $\mathrm{d}\alpha_1/\mathrm{d}a$, . . ., $\mathrm{d}\alpha_k/\mathrm{d}a$, $\mathrm{d}\beta_0/\mathrm{d}a$, $\mathrm{d}\beta_1/\mathrm{d}a$, . . ., $\mathrm{d}\beta_k/\mathrm{d}a$ as functions of $\mathrm{d}A(a^*)/\mathrm{d}a$. Substituting them into the 1st equation of (2.13) we get an algebraic equation containing only one variable $\mathrm{d}A(a^*)/da$. By using Hurwits criterion the sign of $\mathrm{d}A(a^*)/\mathrm{d}a$ can be determined and then the stability of the periodic solution can be determined as well.

If the order of above algebraic equation is higher than five, it is quite complicated to determine the sign of $\mathrm{d}A/\mathrm{d}a$ by using Hurwits criterion. In this case we will use another criterion based on power-energy principle to decide the stability.

Rewrite (2.1) in the form

$$\mathrm{d}(\frac{1}{2}\dot{x}^2 + V(x)) = -f(x, \dot{x})\dot{x}\mathrm{d}x \qquad V(x) = \int_0^x g(x)\mathrm{d}x \tag{2.14}$$

Suppose the total energy of the system is $E(x,\dot{x})=\frac{1}{2}\dot{x}^2+V(x)$ and the total work done by the damping force $f(x,\dot{x})\dot{x}$ along a closed curve $l$ is $W=\oint_l f(x,\dot{x})\dot{x}\mathrm{d}x$, then from (2.14), we get

$$E(x_T,\dot{x}_T)-E(x_0,\dot{x}_0)=-W$$

For the vibration system, the total energy must be positive and correspond to a closed curve in the phase plane. Suppose the initial point $(x_0,\dot{x}_0)$ of the periodic solution is located on close curve $l^*$ (see Fig.1), after a period $2\pi$ it would go back to the original place, i.e. $x(2\pi)=x_0, \dot{x}(2\pi)=\dot{x}_0$, therefore, along $l^*$ we would have $W=\oint_{l^*} f(x,\dot{x})\dot{x}\mathrm{d}x=0$. If the initial point is located on the close curve $l_1$, and $W_1=\oint_{l_1} f(x,\dot{x})\dot{x}\mathrm{d}x<0$, from (2.14) we know that after a period $2\pi$ its end point must be outside $l_1$. Similarly if $(x_0,\dot{x}_0)$ is located on $l_2$ and $W_2=\oint_{l_2} f(x,\dot{x})\dot{x}\mathrm{d}x>0$ after a period $2\pi$ the end point must be inside $l_2$ (see Fig.1). In this case it is obvious that the periodic solution must be stable. So we get the following theorem.

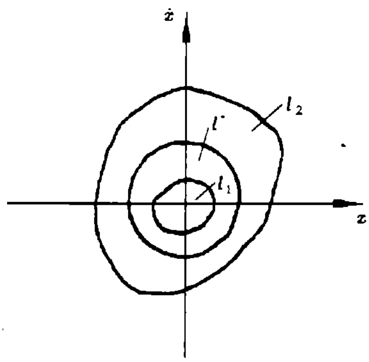

Fig.1

**Theorem**: For system (2.1), if $A(a^*)=0, W(a)<0$ as $a<a^*, W(a)>0$ as $a>a^*$, then the periodic solution corresponding to $a^*$ is stable, otherwise unstable.

## III. PERIODIC SOLUTION OF LIEŃARD EQUATION

Consider Lieńard equation

$$\ddot{x}+\eta(x^2-\gamma)\dot{x}+\alpha x+\beta x^3=0 \qquad \eta\ \gamma\ \alpha \text{ and } \beta \text{ all} >0 \tag{3.1}$$

First we assume that the solution of (3.1) is

$$x=a\cos\theta+b(a) \qquad \frac{\mathrm{d}a}{\mathrm{d}t}=A(a) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t}=B(a) \tag{3.2}$$

Because the potential energy of (3.1) is symmetric, the deviation of vibration center from the origin $O$ must be zero. i.e. $b(a)=0$. In this case (2.6) would be

$$(\frac{\mathrm{d}A}{\mathrm{d}a}-\eta\gamma+\frac{3}{4}\eta a^2)A+a(-B^2+\alpha+\frac{3}{4}\beta a^2)=0 \tag{3.3}$$

$$(-2B-a\frac{\mathrm{d}B}{\mathrm{d}a})A+\eta aB(-\frac{a^2}{4}+\gamma)=0 \tag{3.4}$$

Putting $A=0$ we obtain

$$a^*=2\sqrt{\gamma} \qquad B(a^*)=\sqrt{\alpha+\frac{3}{4}\beta a^*} \tag{3.5}$$

As for (2.13) they are of the form

$$\begin{aligned}&(\frac{\mathrm{d}A(a^*)}{\mathrm{d}a}-\eta\gamma+\frac{3}{4}\eta a_0^2)\frac{\mathrm{d}A(a^*)}{\mathrm{d}a}+a_0(\frac{\mathrm{d}B(a^*)}{\mathrm{d}a}+\frac{3}{2}\beta a_0)=0\\ &a_0\frac{\mathrm{d}B^2(a^*)}{\mathrm{d}a}\frac{\mathrm{d}A(a^*)}{\mathrm{d}a}+4(\alpha+3\beta\gamma)(\frac{\mathrm{d}A(a^*)}{\mathrm{d}a}+\eta\gamma)=0\end{aligned} \tag{3.6}$$

and (3.6) becomes

$$\left(\frac{\mathrm{d}A(a^*)}{\mathrm{d}a}\right)^3 + 2\eta\gamma\left(\frac{\mathrm{d}A(a^*)}{\mathrm{d}a}\right)^2 + 2(2\alpha + 9\beta\gamma)\frac{\mathrm{d}A(a^*)}{\mathrm{d}a} + 4\eta\gamma(\alpha + 3\beta\gamma) = 0 \tag{3.7}$$

Since $\gamma$, $\alpha$ and $\beta$ all $> 0$, by using Hurwits criterion, it is easy to know that $\frac{\mathrm{d}A(a^*)}{\mathrm{d}a} < 0$ as $\eta > 0$, and $\frac{\mathrm{d}A(a^*)}{\mathrm{d}a} > 0$ as $\eta < 0$. Therefore the periodic solution corresponding to $\eta > 0$ is stable, that corresponding to $\eta < 0$ is unstable.

Putting $\eta = 2, \gamma = 1, \alpha = 3, \beta = 5$, the approximate expression of the periodic solution of (3.1) is $x = 2\cos(4.2426t)$. From Fig.2 we can see the curve obtained by above analytical solution coincides with that obtained by numerical integration very well. From Fig.3 we can also see this periodic solution is stable.

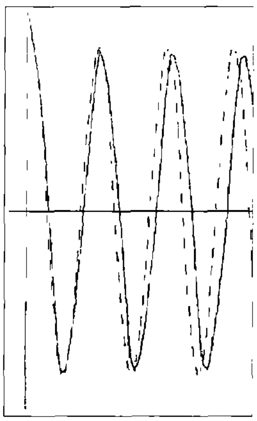

Fig.2 $x$-$t$
- - - analytical; — numerical

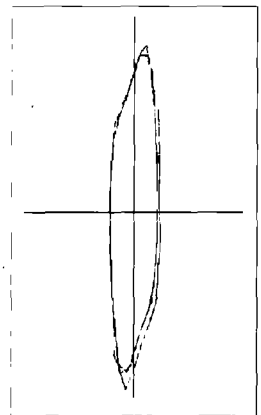

Fig.3 $\dot{x}$-$x$ Limit cycle

Now we further assume that $\mathrm{d}\theta/\mathrm{d}t$ is a function of both amplitude $a$ and phase angle $\theta$

$$\begin{aligned} & x = a\cos\theta \\ & \frac{\mathrm{d}a}{\mathrm{d}t} = A(a) \\ & \frac{\mathrm{d}\theta}{\mathrm{d}t} = B(a,\theta) = B_0(a) + B_1(a)\cos\theta + C_1(a)\sin\theta \\ & \qquad + B_2(a)\cos 2\theta + c_2(a)\sin 2\theta + \cdots \end{aligned} \tag{3.8}$$

For simplicity, here we only consider the components $\sin n\theta$, $\cos n\theta$ ($n \leq 2$) in $\mathrm{d}\theta/\mathrm{d}t$. In this case (2.6) would be of the form

$$\begin{aligned} & A\left(\frac{\mathrm{d}A}{\mathrm{d}a} - 2C_2 - a\frac{\mathrm{d}C_2}{\mathrm{d}a} + \mu\frac{3}{4}a^2 - \mu\gamma\right) + a(-B_0^2 - 2C_1 - 2B_2 - 2C_2 \\ & \quad -\mu\frac{a^2}{2}C_2 + \alpha + \frac{3}{4}\beta\alpha^2) = 0 \end{aligned} \tag{3.9}$$

$$\begin{aligned} & A(-2B_0 + 2B_2 - a\frac{\mathrm{d}B_0}{\mathrm{d}a} + a\frac{\mathrm{d}B_2}{\mathrm{d}a}) + a(2B_1C_1 - \mu\frac{a^2}{4}B_0 \\ & \quad +\mu\gamma B_0 - \mu\gamma B_2) = 0 \end{aligned} \tag{3.10}$$

$$A(2C_1 + a\frac{\mathrm{d}C_1}{\mathrm{d}a}) + a(-3B_0B_1 - 4C_1C_2 + \mu\frac{a^2}{2}C_1 - \mu\gamma C_1) = 0 \tag{3.11}$$

$$A(-a\frac{\mathrm{d}B_1}{\mathrm{d}a} - 2B_1) + a(4C_1B_2 - 2B_0C_1 - \mu\frac{a^2B_1}{2} + \mu\gamma B_1) = 0 \tag{3.12}$$

$$A(2C_2 + a\frac{\mathrm{d}C_2}{\mathrm{d}a} + \mu\frac{a^2}{4}) + a(-4B_0B_2 - 2B_1^2 + 2C_1^2 + B_2^2 - C_2^2$$
$$-\mu\gamma C_2 + \mu\frac{a^2}{4}C_2) = 0 \tag{3.13}$$

$$A(-2B_2 - a\frac{\mathrm{d}B_2}{\mathrm{d}a}) + a(-4B_1a - 4B_0C_2 + 2B_2C_2$$
$$+\mu\gamma B_2 - \mu\frac{a^2}{4}B_0 - \mu\frac{a^2}{4}B_2) = 0 \tag{3.14}$$

(3.9)–(3.14) are very complicated, so we make the following simplifications. First we rewrite $\mathrm{d}\theta/\mathrm{d}t$ as

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = B(a,\theta) = B_0(1 + \frac{B_1}{B_0}\cos\theta + \frac{C_1}{B_0}\sin\theta + \frac{B_2}{B_0}\cos 2\theta + \frac{C_2}{B_0}\sin 2\theta + \cdots) \tag{3.15}$$

For the periodic solution the sign of $\mathrm{d}\theta/\mathrm{d}t$ must be unchanged, hence $|B_1/B_0|$, $|B_2/B_0|$, $|C_1/B_0|$, $|C_2/B_0|$ are all less than one. Therefore compared with $B_0$ the $B_2^2, C_2^2$ and $B_2C_2$ are all small quantities and can be neglected in the first order approximation. Next, from above we know that when $\mathrm{d}\theta/\mathrm{d}t$ is only a function of amplitude $a$, we get $a^* \approx 2\sqrt{\gamma}$ and Fig.2 shows that the difference between the analytical solution and the numerical integration is very small. This means the quantity $\frac{a_0^2}{4} - \gamma$ is very small and the term $\left(\frac{a_0^2}{4} - \gamma\right)^2$ in (3.9)–(3.14) can be neglected. After these simplifications, now we put $A = 0$, from (3.9)–(3.14) it is easy to obtain

$$a^* = 2\sqrt{\gamma - \frac{\beta\gamma^2}{4\alpha + 13\beta\gamma + \frac{3}{4}\mu^2\gamma^2}} \tag{3.16}$$

$$B_0 = \alpha + \frac{3}{4}\beta a^2(a^*) - \frac{\mu^2}{4}\left(\frac{a^2(a^*)}{4} - \gamma\right)\left[-\frac{a^2(a^*)}{4} + \frac{1}{\gamma}\left(\frac{a^2(a^*)}{4} - \gamma\right)^2\right] \tag{3.17}$$

$$B_1 = C_1 = 0 \tag{3.18}$$

$$B_2 = -\frac{1}{\gamma}\left(\frac{a^2(a^*)}{4} - \gamma\right)B_0 \tag{3.19}$$

$$C_2 = -\mu\frac{a^2(a^*)}{16} + \frac{\mu}{4\gamma}\left(\frac{a^2(a^*)}{4} - \gamma\right)^2 \tag{3.20}$$

By calculation, we get

$$x = a_0\left(1 - \frac{B_2}{2B_0}\right)\cos B_0t - \frac{aC_2}{2B_0}\sin B_0t + \frac{aB_2}{6B_0}\cos 3B_0t + \frac{aC_2}{6B_0}\sin 3B_0t \cdots \tag{3.21}$$

The stability of the periodic solution can be decided by Theorem. Since

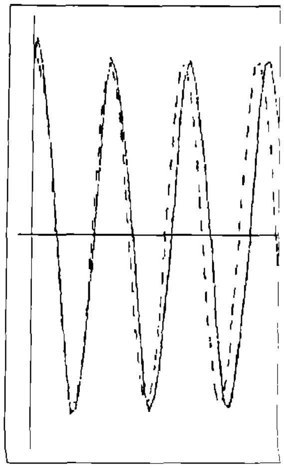

Fig.4 $x$-$t$

- - - analytical; — numerical

$$W(a) = \oint f(x,\dot{x})\dot{x}\mathrm{d}x$$
$$= \oint_0^{2\pi} \eta(a^2\cos^2\theta - \gamma)a\sin\theta(B_0 + B_1\cos\theta$$
$$+C_1\sin\theta + B_2\cos 2\theta + C_2\sin 2\theta)\mathrm{d}\theta$$
$$= \frac{\pi a^2 B_0 \eta}{2}(\frac{a^2}{4} - \gamma)$$

From (3.16) we know $a^* = 2\sqrt{\gamma}$, therefore it is easy to see that $W(a) > 0$ as $a > a^*$, $W(a) < 0$ as $a < a^*$. Then by Theorem the periodic solution is stable. Putting $\eta = 2, \gamma = 1$, $\alpha = 3, \beta = 5$ ($x_0 = 2,\ \dot{x}_0 = 5$), from (3.16)–(3.21) we get

$$x = 1.8762\cos(4.228t) + 0.1069\sin(4.228t)$$
$$+0.0202\cos(3\times 4.228t) - 0.0356\sin(3\times 4.228t) \quad (3.22)$$

The curve obtained by (3.22) and that by numerical intergration are shown in Fig.4.

## IV. COMPARISON WITH THE TRADITIONAL METHOD OF AVERAGING

The authors of [2] pointed out that for some kinds of weakly nonlinear systems the limit cycle can not be obtained by the traditional method of averaging trigonometic functions. But in our method, by averaging trigonometic functions the limit cycle can still be obtained. For the sake of comparison, we use the same example as that in [2].

Consider the nonlinear system[2]

$$\ddot{x} + x = \varepsilon(-x^3 + \frac{1}{2}\dot{x} + \frac{31}{10}x^2\dot{x} - \dot{x}^3) \qquad \varepsilon = \frac{1}{10} \tag{4.1}$$

Suppose the solution of Eq.(4.1) is

$$x = c(t)\cos\psi(t) \tag{4.3}$$

By using the traditional method of averaging, we have

$$\begin{aligned}\frac{\mathrm{d}c}{\mathrm{d}t} &= \frac{\varepsilon}{80}c(c^2 + 20)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= 1 + \frac{3}{8}\varepsilon c^2\end{aligned} \tag{4.4}$$

Put $\mathrm{d}c/\mathrm{d}t = 0$, we get $c = 0$, this means (4.1) has no limit cycle. But the (4.1) really has a limit cycle (see Fig.5). Hence the authors of [2] turn to use the method of averaging elliptic functions and get the approximate expression of the periodic solution $x = 1.9861\mathrm{cn}(1.180t, 0.3760)$. From this we know that the amplitude $a$ is 1.9861, and the frequency is 1.180. As for the stability of the periodic solution it is not discussed in [2].

Now we solve the problem by using the method given in this paper. Suppose the periodic solution of (4.1) is

$$x = a\cos\theta \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = A(a) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = B(a) \tag{4.5}$$

Substituting (4.5) into (4.1) and noticing (2.6) we have

$$A(\frac{\mathrm{d}A}{\mathrm{d}a} - \varepsilon\alpha - \frac{3}{4}\varepsilon\beta a^2 + \frac{3}{4}\varepsilon A^2 + \frac{3}{4}\varepsilon a^2 B^2) - aB^2 + \frac{3}{4}\varepsilon a^3 + a = 0 \tag{4.6}$$

$$A(-2B - a\frac{\mathrm{d}B}{\mathrm{d}a} - \frac{3}{4}\varepsilon aAB) + \varepsilon aB(\alpha + \frac{\beta a^2}{4} - \frac{3}{4}a^2B^2) = 0 \tag{4.7}$$

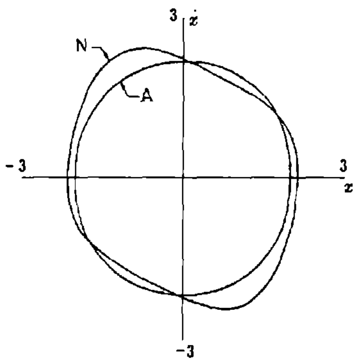


Fig.5
N — numerical;
A — averaging elliptic function

Putting $A = 0$ we get $a^* = 1.792, B = 1.114$. This shows that the amplitude and frequency obtained here are almost the same as that obtained by using the method of averaging elliptic functions.

As for $W(a)$ it can be calculated as follows

$$\begin{aligned} W(a) &= \oint f(x,\dot{x})\dot{x}\mathrm{d}x = \oint \varepsilon(\dot{x}^2 - 3.1x^2 - 0.5)\dot{x}\mathrm{d}x \\ &= \varepsilon Ba^2 \int_0^{2\pi} (a^2B^2\sin^2\theta - 3.1a^2\cos^2\theta - 0.5) \\ &\quad \cdot \sin^2\theta \mathrm{d}\theta \\ &= \frac{\varepsilon a^2 B\pi}{160}(9a_0^4 - 4a_0^2 - 80) \end{aligned}$$

Since $W(a) > 0$ as $a > a^*$, $W(a) < 0$ as $a < a^*$, then by Theorem the periodic solution is stable.

## V. CONCLUSION

The method given here for studying the periodic solution of strongly nonlinear systems is very simple in calculation, because only trigonometric functions are involved. The results obtained by this method agree with those obtained by numerical integration very well. This method can also be applied to strongly nonlinear nonautonomous system. This will be investigated in another paper.

## REFERENCES

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