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New constructing method for WENO schemes

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Abstract

A new method for constructing weighted essentially non-oscillatory (WENO) scheme is proposed. The idea of this method is to combine Henrick's mapping function and the idea of improving the accuracy of WENO-Z scheme one-by-one order. The particular advantage of the new constructing method is that it can improve the accuracy of WENO scheme near discontinuities. Numerical examples show that the new constructing method is very efficient and robust, and the new WENO scheme is more accurate than the original ones.

Keywords: Numerical method, weighted essentially non-oscillatory scheme, shock wave, complex flowfield.

Nomenclature

f	flux function
g	mapping function
h	numerical flux function
p	pressure
t	time
u	unknown variable/velocity
Greek symbols	
ρ	density
ω, ψ	weights
Subscripts	
i ,*k*	indices
Superscripts	
l	index

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1. Introduction

The weighted essentially nonoscillatory(WENO) schemes have been developed and widely used in past two decades. The basic idea of WENO scheme was firstly proposed by Liu et al.[1], in which the smoothest stencil of the ENO schemes[2] is replaced by aconvex combination of the reconstructions on all candidate stencils. In order to obtain higher order accuracy in smooth regions and keep the essentially non-oscillatory property around discontinuities, the design of the weight of each stencil is very important.

Jiang and Shu[3] analyzed that an r thorder ENO scheme can only be converted into an $(r+1)$ th order WENO scheme by using the smoothness indicator introduced byLiu et al.[1]. And then a classic fifth-order WENO scheme with a general framework for designing the smoothness indicators and weights was proposed by Jiang and Shu[3]. Henrick et al.[4] pointed out that the smoothness indicatorsof Jiang and Shu fail to improve the accuracy order of WENO scheme at critical points, where the first derivatives arezero. A mapping function is proposed by Henrick et al. [4] to obtain the optimal order at critical points.Recently, Borgeset al. [5] suggest use the whole 5-points stencil to devise a smoothness indicator of higher order than the classicalsmoothness indicator proposed by Jiang and Shu [3]. On the other hand, a class of higher than 5th order weighted essentiallynon-oscillatory schemes are designed by Balsara and Shu in [6] and by Gerolymos et al. in[7]. Wang and Chen [8] proposed optimizedWENO schemes for linear waves with discontinuity. Martin et al[9] proposed a symmetric WENO methodby means of a new candidate stencil, the new schemes are 2rth-order accurate and symmetric, and lessdissipative than Jiang and Shu's scheme.

Most of the above mentioned WENO schemes are designed to have ($2r-1$)th or $2r$ th[9] order of accuracy in thesmooth regions directly from rth ENO schemes. Their focus is mainly on improving the accuracy in smooth regions, especially at the critical point($f_i^{'}=0$). Hence, for a solution containing discontinuity, these methods cannot obtain the optimal accuracy at transition point, which connects a smooth region and a discontinuity point. For example, Shen and Zha[10] analyzed the existed fifth order WENO schemes, their accuracy at transition point is only the third order. This shortcoming can affect the general performance of the fifth-order WENO schemes, for example, it can results in the reduced accuracy in simulating the local separated flow induced by shock waves, the excessive numerical diffusion in the flows with shock/turbulence interaction.

Since the solution at the transition point is still smooth, ideally, the discretization accuracy of its first-order derivative can reach to fourth order if only if a smooth stencil with five points (notice that, for constructing the conservative numerical flux, the stencil has one point less. That is, a fourth-order numerical flux can be constructed by using a smooth stencil with four points) is used. In [10], Shen and Zha introduced two fourth-order reconstructions combined with an estimation of smoothness/ non-smoothness of two adjacent four-point stencils to improve the accuracy. In early work[11], Shen et al. indicate that the smoothnessindicator IS_k of Jiang and Shu's WENO scheme does not satisfy the condition $\beta_k = D(1+O(\Delta x^2))$ atthe critical point ($f_i^{'}=0$), and proposed a step-by-step reconstruction to avoid the strict condition. But the method does not satisfy the necessary and sufficient conditionsfor fifth-order convergence[4] atcritical point.

In this paper, based on the analysis[10], a new method for constructing weighted essentially non-oscillatory (WENO) scheme is proposed. The idea of this method is to combine Henrick's mapping function and the idea of improving the accuracy of WENO-Z scheme one-by-one order. The particular advantage of the new constructing method is that it can improve the accuracy of WENO scheme near discontinuities without reducing the accuracy in smooth regions. Numerical examples show that the new constructing method is very efficient and robust, and the new WENO schemes is more accurate than the abovementioned ones.

2. The numerical algorithm

For the hyperbolic conservation law in the form

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \qquad (1)$$

the flux function $f(u)$ can be split into two parts as $f(u)=f^{+}(u)+f^{-}(u)$ with $df^{+}(u)/du\geq 0$ and $df^{-}(u)/du\leq 0$. The semi-discretizationform of (1) can be written as

$$\frac{du_i(t)}{dt}=-\frac{1}{\Delta x}(h_{i+1/2}-h_{i-1/2}) \qquad (2)$$

where the numerical flux is $h_{i+1/2}=h^{+}_{i+1/2}+h^{-}_{i+1/2}$. In this paper, only the positive part $h^{+}_{i+1/2}$ is described and the superscript"+" is dropped for simplicity. The $h^{-}_{i+1/2}$ is evaluated following the symmetric rule about $x_{i+1/2}$.

2.1. Weighted essentially non-oscillatory (WENO) scheme[3-5]

The flux of the fifth-order WENO scheme can be written as

$$h_{i+1/2}=\sum_{k=0}^{2}\omega_k q_k \qquad (3)$$

where q_k is a third-order flux on stencil $S_k^3=(x_{i+k-2},x_{i+k-1},x_{i+k})$, the weight ω_k is constructed as

$$\omega_k=\frac{\alpha_k}{\alpha_0+\alpha_1+\alpha_2}, \text{ with } \alpha_k=\frac{c_k}{(\varepsilon+IS_k)^p}. \qquad (4)$$

IS_k is a smoothness indicator on stencil S_k^3. In [3], Jiang and Shu proposed IS_k as

$$IS_k=\sum_{l=1}^{2}\int_{x_{i-1/2}}^{x_{i+1/2}}\Delta x^{2l-1}[q_k^{(l)}(x)]^2\,dx \qquad (5)$$

$c_0=0.3$, $c_1=0.6$, and $c_2=0.1$ are the optimal weights which generate the fifth-order central upstream scheme. If $f_i^{'}=0$, Eq. (5) gives $IS_k=D(1+O(\Delta x))$ and $\omega_k=c_k+O(\Delta x)$, this will degrade the convergence accuracy of the scheme[4,5,11].

Henrick et al. [4] implemented a detailed truncation error analysis of Jiang and Shu's WENO scheme, and gave the necessary and sufficient conditions for fifth-order convergence of WENO scheme as the following,

$$\begin{cases}\sum_{k=0}^{2}A_k(\omega_k^{+}-\omega_k^{-})=O(\Delta x^3)\\ \omega_k^{\pm}-c_k=O(\Delta x^2)\end{cases}, \qquad (6)$$

where A_k is the coefficient of third-order term (Δx^3) in the Taylor series expansion of q_k to the fifth-order central upstream approximation[4]. To improve the accuracy of weights ω_k, a mapping function $g_k(\omega)$ is defined in [4]as

$$g_k(\omega)=\frac{\omega(c_k+c_k^2-3c_k\omega+\omega^2)}{c_k^2+\omega(1-2c_k)}, \qquad (7)$$

and an improved WENO scheme (WENO-M) is constructed by using $g_k(\omega)$ to generate new weights. WENO-M obtained fifth-order convergence at critical points.

Borges et al.[5] proposed a sufficient condition for the fifth-order WENO scheme,

$$\omega_k^{\pm} - c_k = O(\Delta x^3), \quad (8)$$

and introduced a parameter τ_5 as $\tau_5 = | IS_0 - IS_2 |$ to construct the new smoothness indicator IS_k^z as the following,

$$IS_k^z = \frac{IS_k + \varepsilon}{IS_k + \tau_5 + \varepsilon}. \quad (9)$$

Using the IS_k^z to construct the WENO scheme(called as WENO-Z), the new weights can satisfy the sufficient condition Eq.(8) at critical points.

In all formula, the parameter ε is used to avoid the division by zero, $\varepsilon = 10^{-6}$ is used in [3] and $\varepsilon = 10^{-40}$ is used in [4,5]. p is chosen to increase the difference of scales of distinct weights at non-smooth parts of the solution.

2.2. New constructing method for WENO scheme

The analysis of fifth-order WENO schemes of Shen and Zha[10] shows that the accuracy of fifth-order WENO scheme is reduced at the transition point（point $i+1$ in Fig.1）from smooth region to discontinuous point and viceversa.Shen and Zhaproposed a reconstructions method by using an estimation of smoothness/ non-smoothness of two adjacent four-point stencils to improve the accuracy.

In this paper, a new constructing method for WENO scheme is proposed by combining Henrick's mapping function[4] and the idea of improving the accuracy of WENO-Z scheme one-by-one order[11]. Fig. 2 can be used to illustrate the method.

First step, two fourth-order weighted fluxes are constructed as the following.

$$\begin{cases} h_0^4 = \omega_0^{4,0} q_0 + \omega_1^{4,0} q_1 \\ h_1^4 = \omega_0^{4,1} q_1 + \omega_1^{4,1} q_1 \end{cases} \quad (10)$$

The weights $\omega_k^{4,l}$ ($k = 0,1$; $l = 0,1$) is calculated by combining the method of WENO-Z scheme and mapping function,

$$\omega_k^{4,l} = \frac{g_k(\psi_k^{4,l})}{\sum_k g_k(\psi_k^{4,l})}, \ \psi_k^{4,l} = \frac{\alpha_k^{4,l}}{\sum_k \alpha_k^{4,l}}, \ \alpha_k^{4,l} = c_k^{4,l}\left(1 + \frac{\tau_4^l}{IS_{l+k} + \varepsilon}\right), \ \tau_4^l = | IS_{l+1} - IS_l |, \quad (11)$$

and $c_0^{4,0} = 0.25$, $c_1^{4,0} = 0.75$; $c_0^{4,1} = 0.5$, $c_1^{4,1} = 0.5$.

Second step, the final fifth-order weighted flux is obtained as the following

$$h_{i+1/2} = \omega_0 h_0^4 + \omega_1 h_1^4 \quad (12)$$

where $\omega_k = \dfrac{g_k(\psi_k)}{\sum_k g_k(\psi_k)}$, $\psi_k = \dfrac{\alpha_k}{\sum_k \alpha_k}$, $\alpha_k = c_k^5\left(1 + \dfrac{\tau_5}{IS_{2k} + \varepsilon}\right)$, $\tau_5 = | IS_2 - IS_0 |$, and $c_0^5 = 0.4$, $c_1^5 = 0.6$.

Now, let's analyze the accuracy of the new method (10) and (12). Using Taylor expansion, there is

$$\tau_4^l = | f_i^{'} f_i^{'''} | \Delta x^4 + O(\Delta x^5), \tag{13}$$

so if $f_i^{'} \neq 0$, then $\psi_k^{4,l} = c_k^{4,l} + O(\Delta x^2)$; if $f_i^{'} = 0$ and $f_i^{''} \neq 0$, then $\psi_k^{4,l} = c_k^{4,l} + O(\Delta x)$. Applying the property of mapping function $g_k(\omega)$

$$g_k(\psi_k^{4,l}) = c_k^{4,l} + \frac{(\psi_k^{4,l} - c_k^{4,l})^3}{c_k^{4,l} - (c_k^{4,l})^3} + \cdots, \tag{14}$$

it is easy to find that

$$\omega_k^{4,l} = c_k^{4,l} + o(\Delta x^6) \text{ or } \omega_k^{4,l} = c_k^{4,l} + o(\Delta x^3) \tag{15}$$

Similarly, there is

$$\omega_k = c_k^5 + o(\Delta x^9) \text{ or } \omega_k = c_k^5 + o(\Delta x^3) \tag{16}$$

That means, in above two steps, the sufficient condition of (8) is always satisfied no matter whether $f_i^{'} \neq 0$ or $f_i^{'} = 0$ and $f_i^{''} \neq 0$. Hence there is no accuracy reducing in the multi-step process, the method of (10) and (12) is fifth-order accuracy in smooth regions.

If x_i is a transition point (for example, the discontinuity is between x_{i+1} and x_{i+2}), then in the first step, the fourth-order flux h_0^4 is obtained from Eq.(10). In the second step, the final flux $h_{i+1/2}$ is approximated as $h_{i+1/2} \to h_0^4$ due to $IS_2 >> IS_1$ and $\omega_1 \to 0$ in Eq. (12). Hence the fourth-order accuracy at the transition point is obtained.

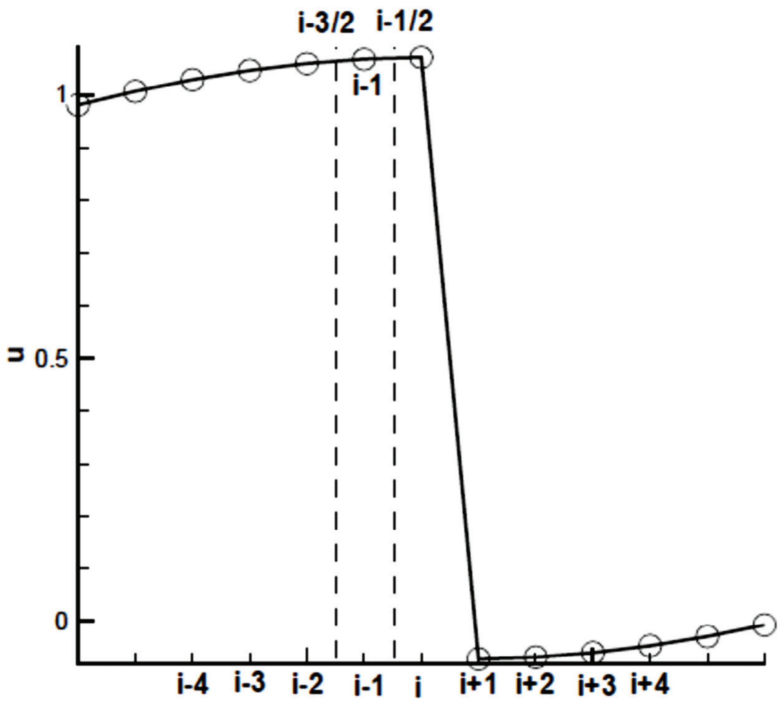

Fig.1 The sketch of transition point

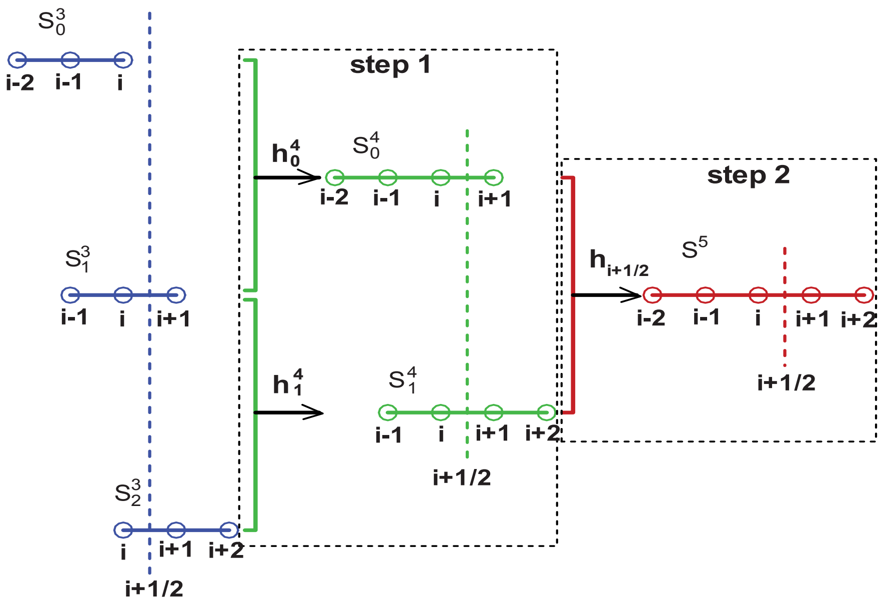

Fig.2 Multi-step constructing process

3. Numerical examples

In this paper, the 4th order Runge-Kutta-type method[12] is used for the time integration.

3.1. Linear transport equation

The linear transport problems are controlled by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0, \quad -1 \le x \le 1 \\ u(x,0) = u_0(x), \;\; \textit{periodic boundary} \end{cases} \tag{17}$$

(1) Initial solution $u_0(x) = \sin\left(\pi x - \dfrac{\sin(\pi x)}{\pi}\right)$

Table 1 gives the errors and accuracy order. It can be seen that, for the smooth solution, the presentscheme obtains the same fifth-order accuracy as WENO-M and WENO-Z schemes.

Table 1.Comparison of accuracy, T=2.

Scheme	N	L_1 error	L_1 order	L_∞ error	L_∞ order
WENO-Z	40	0.217102e-3	-	0.677211e-4	-
	80	0.649393e-5	5.063	0.237405e-5	4.834
	160	0.204882e-6	4.986	0.785200e-7	4.918
	320	0.748874e-8	4.774	0.250232e-8	4.971
	640	0.364893e-9	4.359	0.779779e-10	5.004
WENO-M	40	0.210766e-3	-	0.672781e-4	-
	80	0.648426e-5	5.023	0.225867e-5	4.897
	160	0.204671e-6	4.986	0.720345e-7	4.971
	320	0.640983e-8	4.997	0.226830e-8	4.989
	640	0.200631e-9	4.998	0.710974e-10	4.996
present	40	0.203332e-3	-	0.714827e-4	-
	80	0.649369e-5	4.969	0.229242e-5	4.963
	160	0.204635e-6	4.988	0.724031e-7	4.985
	320	0.640982e-8	4.997	0.227140e-8	4.994
	640	0.200642e-9	4.998	0.711126e-10	4.997

(2) Initial solution $u_0(x) = \begin{cases} -\sin(\pi x) - \frac{1}{2}x^3, & -1 < x \le 0 \\ -\sin(\pi x) - \frac{1}{2}x^3 + 1, & 0 < x \le 1 \end{cases}$

Fig. 3 shows the numerical solutions at t = 6. It can be seen that, near the discontinuity, the presentmethod obtains more accurate solution than WENO-Z and WENO-M schemes.

3.2. Nonlinear transport equation

The nonlinear transport equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad 0 \le x \le 2\pi \tag{18}$$

is solved with initial and boundary conditions:

$$u_0(x) = 0.3 + 0.7\sin(x)\,, 0 \le x \le 2\pi \text{, periodic boundary.}$$

The flux splitting $f^{\pm} = (f \pm au)/2$ is applied, where $f = u^2/2$ and $a = \max(u_i)$. Fig. 4 shows the results at t = 2 with grid number of N = 80. It can be seen that, near the shock, thesolution calculated by the present scheme is closer to the discontinuous solution than WENO-Z and WENO-M schemes.

3.3. One dimensional shock tube problems

The one dimensional Euler equations of gas dynamics is solved. The first-order global Lax-Friedrichsflux[5,6] isused as the low-order building block for the high-order reconstruction of various WENO schemes.

(1) Sod problem

The initial conditions are

$$(\rho, u, p) = \begin{cases} (1, 0, 1), & x < 0 \\ (0.125,\ 0,\ 0.1), & x \ge 0 \end{cases} \tag{19}$$

The solution at $t = 0.14$ is given in Fig. 5. It can be seen that, near shocks, the present method is more accurate than both WENO-Z and WENO-M schemes.

(2) Shu-Osher problem

The initial conditions are

$$(\rho, u, p) = \begin{cases} (3.857143,\ 2.629369,\ 10.33333), & \text{when } x < -4 \\ (1 + \varepsilon \sin 5x, 0, 1), & \text{when } x \geq -4 \end{cases} \tag{20}$$

This case [13] represents a Mach 3 shock wave interacting with a sine entropy wave. The results at time $t = 1.8$ are plotted in Fig. 6. The 'exact' solutions are the numerical solutions of WENO-Z scheme with grid points of $N = 2000$. It can be seen that, even in thesmooth region, the present scheme are more accurate than WENO-Z and WENO-M schemes. This indicates that,if the solution varies dramatically, the new method is lesser dissipative than the other two schemes.

4. Conclusion

By combining Henrick's mapping function and the idea of improving the accuracy of WENO-Z scheme one-by-one order,a new method for constructing weighted essentially non-oscillatory (WENO) scheme is developed. In each step of weighting process, the sufficient condition for fifth-order convergence is kept, hence the final scheme can obtain the fifth-order accuracy in smooth regions even containing critical points. The particular advantage of the new scheme is that it improves the accuracy of WENO scheme at transition points, hence its numerical dissipation near discontinuities is smaller than other fifth-order WENO schemes. Numerical examples show that the new scheme is efficient, robust, and accurate.

Acknowledgements

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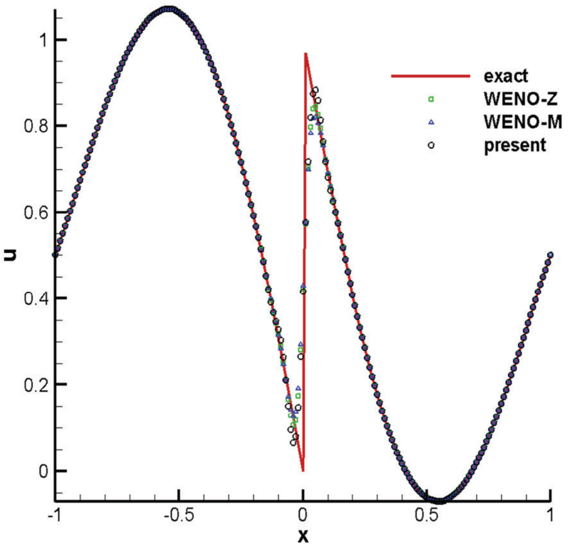

Fig. 3 Numerical results, t=6

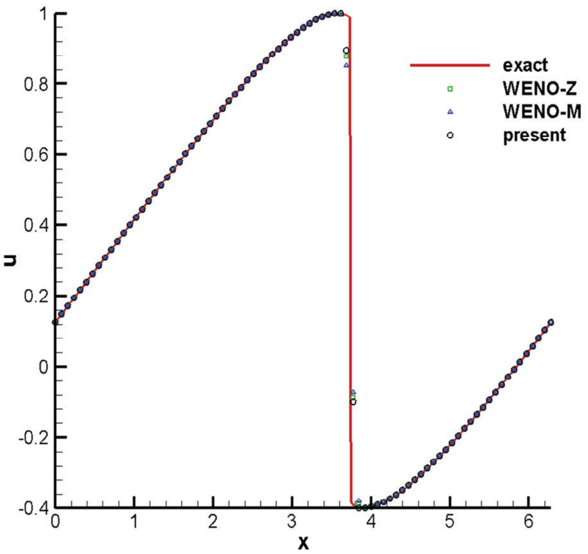

Fig. 4 Numerical results, t=2

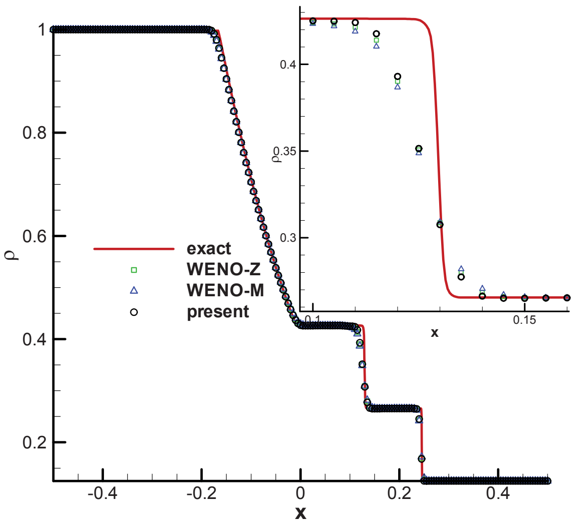

Fig.5 Density, Sod problem

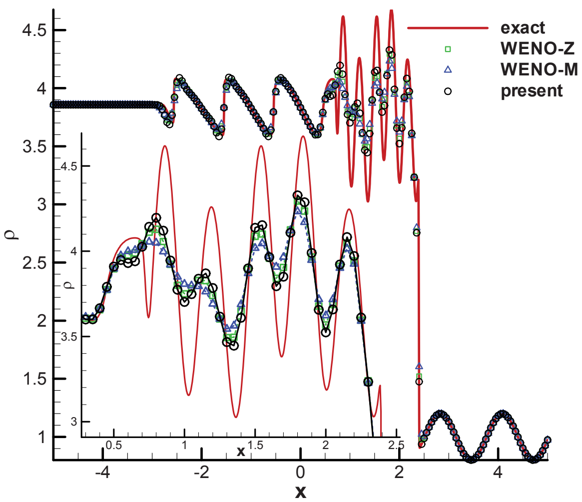

Fig.6 Density, Shu-Osher problem