



上海交通大学
SHANGHAI JIAO TONG UNIVERSITY



非线性力学国家重点实验室年会

统一波浪模型 (UWM) ——挑战传统观念

廖世俊

海洋工程国家重点实验室

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1. 研究动机
2. 统一波浪模型
3. 光滑波浪与尖峰孤立波之统一
4. 有限水深尖峰孤立波之独特性质
5. 畸形波的一种新理论解释
6. 结论和展望



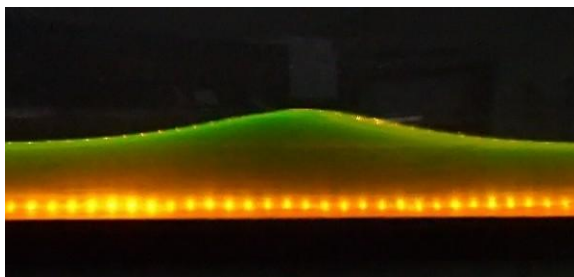
(1) 研究动机

传统的深水波模型和浅水波模型均为精确波浪方程之特例，他们都可以给出光滑的、无限可微的波面。

(a) 浅水波：周期波、孤立波

KdV方程， Boussinesq方程， BBM方程

光滑孤立波：



(b) 有限水深波：椭圆余弦波

精确的波浪方程



(1) 研究动机

传统光滑波浪之性质

1. 波面、流速处处光滑、无限可微
2. 从自由液面到底部，微团速度呈指数衰减
3. 波高越大，波速越大 (*dispersive wave*)



(1) 研究动机

- 1993年 Camass 和 Holm 在 Physical Review Letters 上发表浅水波模型：

$$u_t + 2\omega u_X - u_{XXt} + 3uu_X = 2u_X u_{XX} + uu_{XXX},$$

首次导出**不光滑**的尖峰 (peaked) 孤立波

$$u(X, t) = c \exp(-|X - ct|),$$

并引发不光滑尖峰孤立波研究的**热潮**

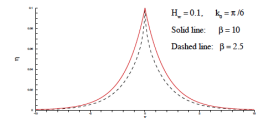


FIG. 2. Peaked solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/2}/\sqrt{\pi}$). Solid line: $\beta = 10$; Dashed line: $\beta = 2.5$.

- In 1999, Kraenkel, R.A. & Zenchuk, A. (J. Phys. A., 32:4733) found the **cusped** solitary waves., whose 1st-derivative tends to **infinity** at crest.

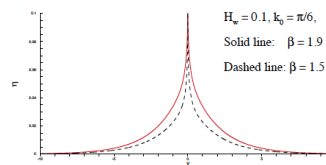


FIG. 1. Cusped solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/4}/\sqrt{\pi}$). Solid line: $\beta = 1.9$; Dashed line: $\beta = 1.5$.



(1) 研究动机

- 所有这些尖峰孤立波，都是由近似的浅水波浪模型导出至今没有从精确波浪方程中导出有限水深中的尖峰孤立波。

这导致疑问：他们理论上是否合理？

- 尖峰孤立波除波峰处不光滑外，还有其他独特性质吗？
例如：流体微团速度、动能在垂向之分布？
- Relationship** between peaked/cusped solitary waves has **never** been reported.

Peaked 孤立波与 cusped 孤立波有何联系？



(2) 统一波浪模型

(1) 假设在**半无限域** $0 < x < +\infty$ ，流体是理想、不可压，流动是无旋的，表面张力可忽略，因此可采用精确波浪方程描述：

控制方程：
$$\nabla^2 \phi(x, z) = 0, \quad z \leq \zeta(x), 0 < x < +\infty,$$

自由面条件：
$$\alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} - \alpha \frac{\partial}{\partial x} (\nabla \phi \cdot \nabla \phi) + \nabla \phi \cdot \nabla \left(\frac{1}{2} \nabla \phi \cdot \nabla \phi \right) = 0, \quad 0 < x < +\infty,$$

$$\zeta - \alpha \frac{\partial \phi}{\partial x} + \frac{1}{2} \nabla \phi \cdot \nabla \phi = 0, \quad 0 < x < +\infty,$$

底部条件：
$$\frac{\partial \phi}{\partial z} = 0, \quad z = -1, 0 < x < +\infty,$$

左边界条件：
$$u(0, z) = \lim_{x \rightarrow 0} \frac{\partial \phi}{\partial x} = U(z), \quad z \leq \zeta(x),$$

右边界条件：

周期波： $\phi(x, z) = \phi(x + \lambda, z)$ ，孤立波： $\phi(\pm\infty, z) = 0$ ，

***** 统一波浪模型容许波峰处 ($x = 0$) 有旋!**



(2) 统一波浪模型

(2) 对称性:

$$\zeta(x) = \zeta(-x), \quad u(x, z) = u(-x, z), \quad v(x, z) = -v(-x, z),$$

将解扩展到整个流域 $-\infty < x < +\infty$.

(3) 由对称性条件, 导出限制条件:

$$v(0, z) = -v(0, z), \quad \text{i.e. } v(0, z) = 0.$$

半无限区域内的拉普拉斯方程:

$$\nabla^2 \phi(x, z) = 0, \quad z \leq \zeta(x), 0 < x < +\infty,$$

具有两种（满足底部边界条件）的解:

(1) 传统的光滑基函数 \longrightarrow 传统的光滑波浪

$$\cosh[nk(z + 1)] \sin(nkx), \quad n \geq 1,$$

(2) 非光滑基函数 \longrightarrow 全新的尖峰孤立波

$$\cos[nk(z + 1)] \exp(-nkx), \quad n \geq 1, k > 0, 0 \leq x < +\infty,$$



(A) 光滑波浪（势函数）之基函数

$$\cosh[nk(z+1)] \sin(nkx), \quad n \geq 1,$$

自动满足对称性条件

$$\zeta(x) = \zeta(-x), \quad u(x, z) = u(-x, z), \quad v(x, z) = -v(-x, z),$$

和限制性条件：

$$v(0, z) = 0.$$

因此，在半无限域 $0 < x < +\infty$ 内得到的所有结果，自动地在整个无穷域内 $-\infty < x < +\infty$ 存立！

结论：统一波浪模型可以给出有限水深中的椭圆余弦波等所有的传统光滑波浪



(B) 非光滑波浪（势函数）之基函数

$$\cos[nk(z+1)] \exp(-nkx), \quad n \geq 1, \quad k > 0, \quad 0 \leq x < +\infty,$$

即不能自动满足对称性条件

$$\zeta(x) = \zeta(-x), \quad u(x, z) = u(-x, z), \quad v(x, z) = -v(-x, z),$$

也不能自动满足限制性条件：

$$v(0, z) = 0.$$

因此，在半无限域 $0 < x < +\infty$ 内得到的所有结果，必须用对称性条件扩展到在整个无穷域内 $-\infty < x < +\infty$ 。

而且，限制性条件 $v(0, z) = 0$ 必须强制满足。



(3) 光滑波浪与尖峰孤立波之统一

有限水深尖峰孤立波之线性解

$$\alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} = 0, \quad \text{on } z = 0, 0 < x < +\infty,$$

$$\zeta(x) = \alpha \left. \frac{\partial \phi}{\partial x} \right|_{z=0}, \quad 0 < x < +\infty,$$

速度势:

$$\phi^+(x, z) = \alpha A \cos[k(z+1)] e^{-kx}, \quad 0 < x < +\infty,$$

波速:

$$\alpha^2 = \frac{\tan k}{k}, \quad n\pi < k < n\pi + \frac{\pi}{2},$$

给定无因次波速, 上述方程有无穷多个解:

$$K_\alpha = \left\{ k_n : \alpha^2 = \frac{\tan k_n}{k_n}, n\pi < k_n < n\pi + \frac{\pi}{2}, n = 0, 1, 2, 3, \dots \right\}$$

(3) 光滑波浪与尖峰孤立波之统一

有限水深尖峰孤立波之线性解

速度势：
$$\phi^+(x, z) = \alpha A_\nu \cos[k_\nu(z+1)] \exp(-k_\nu x), \quad 0 < x < +\infty.$$

波面：
$$\zeta(x) = H_w e^{-k_\nu |x|}, \quad -\infty < x < +\infty.$$

水平速度：
$$\frac{u}{U_0} = \frac{\cos[k_\nu(z+1)]e^{-k_\nu|x|}}{\cos(k_\nu)}, \quad x \in (-\infty, +\infty),$$

垂向速度：
$$v^+(x, z) = \frac{\partial \phi^+}{\partial z} = \frac{\alpha k_\nu H_w \sin[k_\nu(z+1)]e^{-k_\nu x}}{\sin k_\nu}.$$

$$v^-(x, z) = -v^+(-x, z) = -\frac{\alpha k_\nu H_w \sin[k_\nu(z+1)]e^{k_\nu x}}{\sin k_\nu}, \quad -\infty < x < 0$$

(3) 光滑波浪与尖峰孤立波之统一

有限水深尖峰孤立波之非线性解

(1) 求解方法:

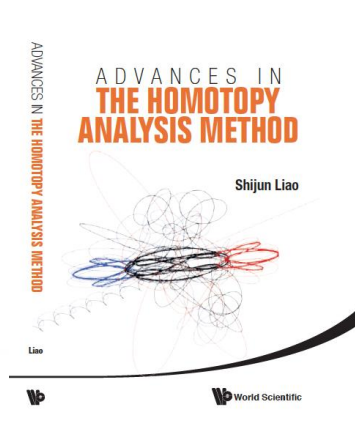
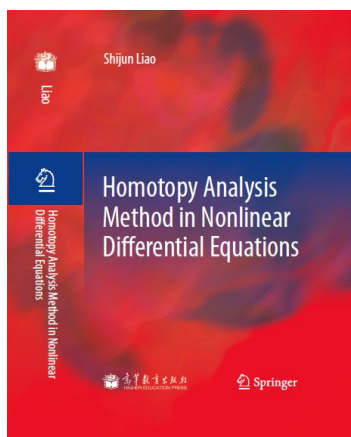
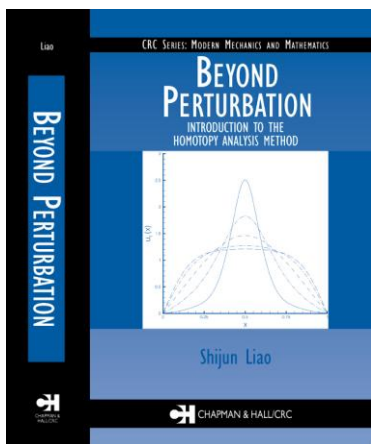
同伦分析方法 (1992, homotopy analysis method, HAM)

优点: 适合于强非线性问题

(a) 不依赖物理小参数;

(b) 确保级数解之收敛;

(c) 可自由选取高阶近似方程的类型及其解的基函数





有限水深尖峰孤立波之非线性解

应用同伦分析方法和光滑基函数，成功获得平衡态的、光滑共振波系（有限水深）及其多解：

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On the steady-state fully resonant progressive waves in water of finite depth

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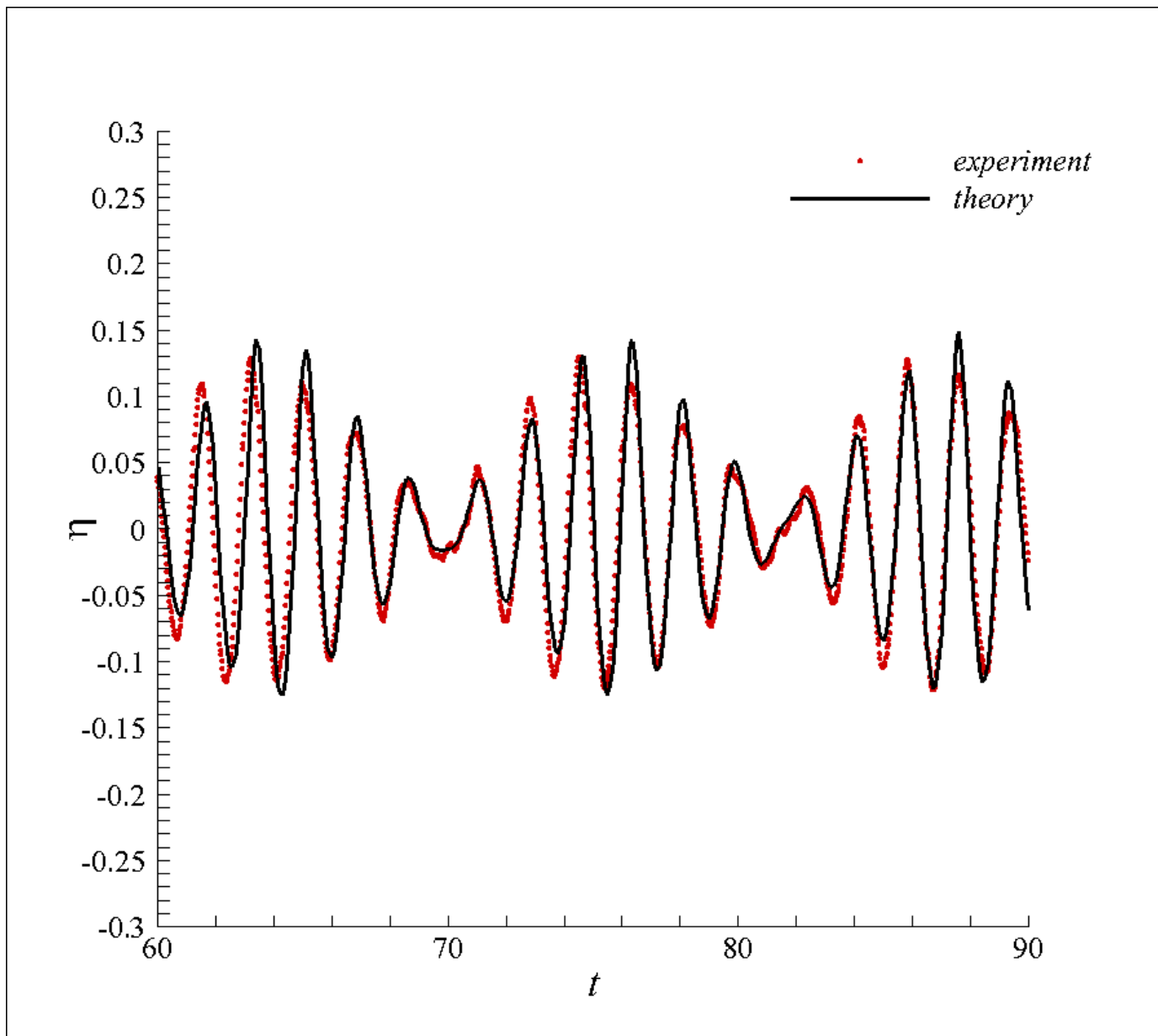
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实验证明：定常共振波系存在性





(3) 光滑波浪与尖峰孤立波之统一

有限水深尖峰孤立波之非线性解

应用同伦分析方法和非光滑基函数，成功获得收敛的级数解！

波面：

波速与波高无关！

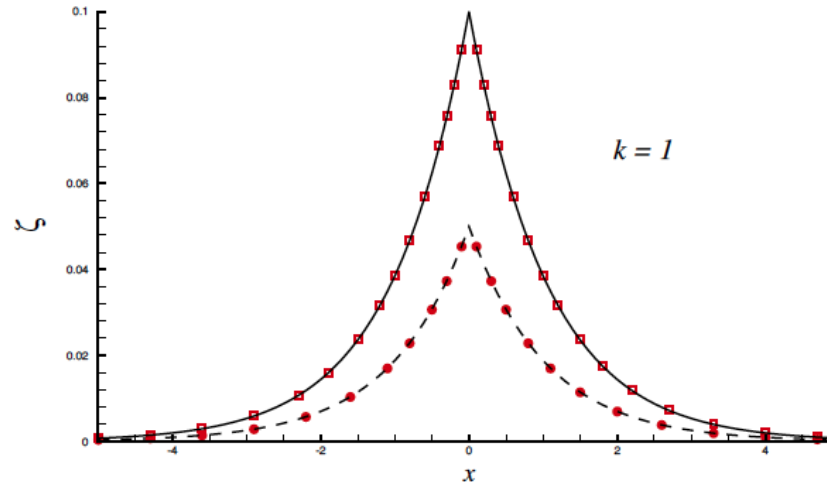


Figure 8: Analytic approximations of elevation of the peaked solitary waves when $k = 1$ (corresponding to $c/\sqrt{gD} = 1.24796$). Solid line: 5th-order approximation when $H_w = 0.1$ given by $c_\phi = -0.5$ and $c_\eta = -1$; Filled circles: 25th-order approximation when $H_w = 0.1$ given by $c_\phi = -0.5$ and $c_\eta = -1$; Dashed line: 5th-order approximation when $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$; Open circles: 25th-order approximation when $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$.



传统光滑波浪

1. 波面和速度处处光滑，无限可微
2. 从波面到水底，流体微团水平速度和动能都呈指数衰减
3. 波速与波高有关：高的波浪传播快

有限水深尖峰孤立波

1. 波峰处波面不光滑，速度间断
2. 从波面到水底，流体微团水平速度增加，动能增加（或略微减小）
3. 波速与波高无关，微小波高的波速可以很大！

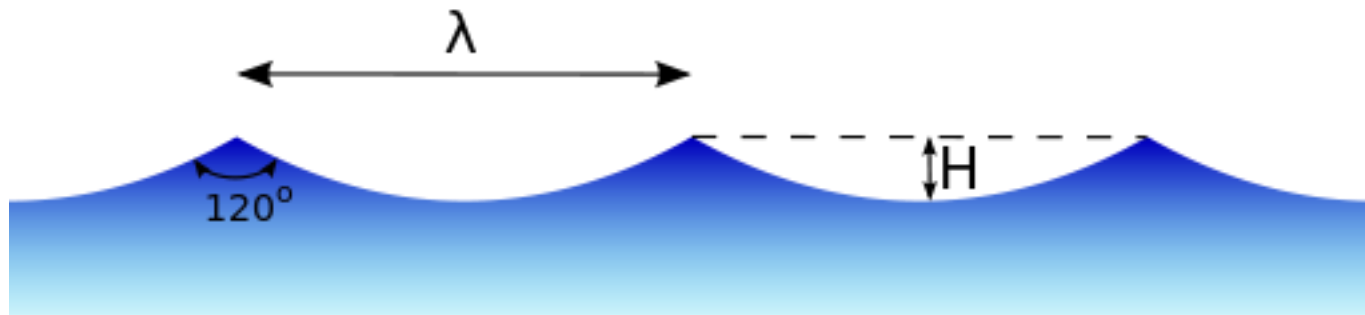
有限水深尖峰孤立波上述独特性质，从未见报道！



传统波浪理论框架容许尖波峰的存在!

G.G. Stokes (1894年) 证明:

光滑的、有限水深中的周期行进波，在极限波高时，其波峰呈120度尖峰(corner crest), 从而不光滑。



结论:

有限水深尖峰孤立波波峰之不光滑，同样也是容许的!



(3) 光滑波浪与尖峰孤立波之统一

有限水深尖峰孤立波之线性解

在传统的波浪理论框架内，光滑界面波交界处，其法向速度连续，但切向速度方向相反！

Lamb (Hydrodynamics, page 371):

“the tangential velocity changes sign as we cross the surface”,
but “in reality the discontinuity, if it could ever be originated,
would be immediately abolished by viscosity”



结论：有限水深尖峰孤立波垂向速度的不连续，同样是可以接受的、合理的！



对有限水深尖峰孤立波，开尔文定理处处成立：

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{r} = \int_{C_1} \mathbf{u}_1 \cdot d\mathbf{r} + \int_{C_2} \mathbf{u}_2 \cdot d\mathbf{r},$$

$$\begin{aligned} \frac{d\Gamma}{dt} &= \int_{C_1} \left\{ \frac{d\mathbf{u}_1}{dt} \cdot d\mathbf{r} + \mathbf{u}_1 \cdot d\mathbf{u}_1 \right\} + \int_{C_2} \left\{ \frac{d\mathbf{u}_2}{dt} \cdot d\mathbf{r} + \mathbf{u}_2 \cdot d\mathbf{u}_2 \right\} \\ &= \left(\frac{p}{\rho_1} + gz + \frac{|\mathbf{u}_1|^2}{2} \right) \Big|_{Q_1}^{Q_2} + \left(\frac{p}{\rho_2} + gz + \frac{|\mathbf{u}_2|^2}{2} \right) \Big|_{Q_2}^{Q_1} \\ &= [p(Q_2) - p(Q_1)] \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \\ &\quad + \frac{1}{2} \{ (|\mathbf{u}_1(Q_2)|^2 - |\mathbf{u}_2(Q_2)|^2) - (|\mathbf{u}_1(Q_1)|^2 - |\mathbf{u}_2(Q_1)|^2) \} \\ &= 0 \end{aligned}$$



On cusped solitary waves in finite water depth

Shijun Liao

(Submitted on 23 Nov 2013)

It is well-known that the Camassa–Holm (CH) equation admits both of the peaked and cusped solitary waves in shallow water. However, it was an open question whether or not the exact wave equations can admit them in finite water depth. Besides, it was traditionally believed that cusped solitary waves, whose 1st-derivative tends to infinity at crest, are essentially different from peaked solitary ones with finite 1st-derivative. Currently, based on the symmetry and the exact water wave equations, Liao [1] proposed a unified wave model (UWM) for progressive gravity waves in finite water depth. The UWM admits not only all traditional smooth progressive waves but also the peaked solitary waves in finite water depth: in other words, the peaked solitary progressive waves are consistent with the traditional smooth ones. In this paper, in the frame of the linearized UWM, we further give, for the first time, the cusped solitary waves in finite water depth, and besides reveal a close relationship between the cusped and peaked solitary waves: a cusped solitary wave is consist of an infinite number of peaked solitary ones with the same phase speed, so that it can be regarded as a special peaked solitary wave. This also well explains why and how a cuspon has an infinite 1st-derivative at crest. It is found that, like peaked solitary waves, the vertical velocity of a cusped solitary wave in finite water depth is also discontinuous at crest ($x=0$), and especially its phase speed has nothing to do with wave height, too. All of these would deepen and enrich our understandings about the cusped solitary waves.



Cusped solitary waves

$$\eta(x) = \frac{H_w}{\zeta(\beta)} \sum_{n=1}^{+\infty} \frac{1}{n^\beta} \exp(-k_{n-1}|x|), \quad \beta > 1, \quad \alpha^2 = \frac{\tan k_n}{k_n}$$

Peaked solitary waves: $\beta > 2$

Cusped solitary waves: $1 < \beta \leq 2$

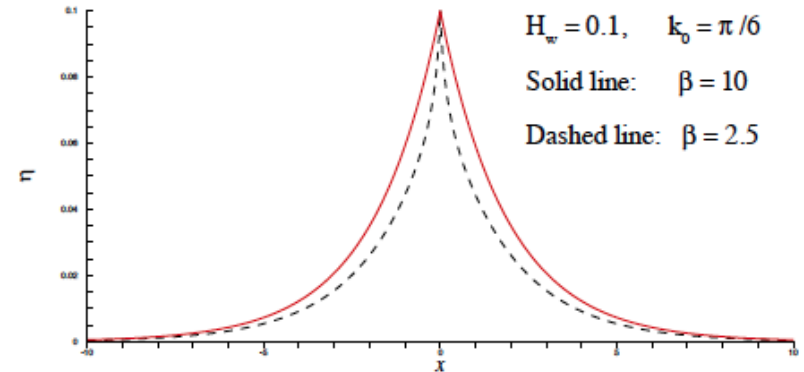
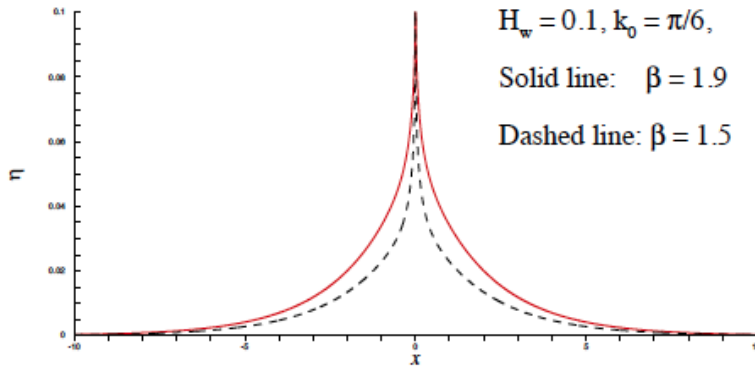


FIG. 1. Cusped solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/4}/\sqrt{\pi}$). Solid line: $\beta = 1.9$; Dashed line: $\beta = 1.5$.

FIG. 2. Peaked solitary waves in finite water depth defined by (6) when $H_w = 0.1$ and $k_0 = \pi/6$ (corresponding to $\alpha = 12^{1/4}/\sqrt{\pi}$). Solid line: $\beta = 10$; Dashed line: $\beta = 2.5$.

Cusped solitary waves have the **same** properties with the peaked ones!

首次揭示了两种尖峰孤立波之关系!



(5) 畸形波的新理论解释

目前对畸形波的理论解释，通常建立在传统光滑波浪的非线性相互作用等基础上。但他们无法解释如下事实：

The rogue wave can **suddenly** appear on ocean even when “**the weather was good, with clear skies and glassy swells**”, as reported by Graham (2000) and Kharif (2003).

畸形波的全新理论解释：

有限水深尖峰孤立波，其波速与波高无关，且微小的尖峰孤立波可以具有很大的波速。许多这些微小的、以不同波速传播的尖峰孤立波，理论上有可能在某处突然叠加成一个巨大的波浪，而其他处很平坦。



(6) 结论和展望

- 建立了“统一波浪模型” (Unified Wave Model, UWM)
- 首次获得两种有限水深尖峰孤立波，揭示其密切联系
- 首次证明：
有限水深尖峰孤立波与传统光滑波浪
在理想流体框架内是相容的，一样合理的
- 发现有限水深尖峰孤立波一些独特的性质
- 给出畸形波全新的理论解释



- 1845: 光滑孤立波 (J. S. Russell)
- 1872: Boussinesq 浅水波方程 (J. Boussinesq)
- 1894: 周期行进波在极限波高下波峰呈尖角 (G.G. Stokes)
- 1895: KdV 浅水波方程 (D. J. Korteweg and G. de Vries)
- 1970s: 有限水深椭圆余弦波 (J.D. Fenton)
- 1993: 浅水尖峰孤立波 (Camassa – Holm 方程)
- 2013: 有限水深尖峰孤立波 (统一波浪模型)

“统一波浪模型” (Unified Wave Model, UWM) 可以给出上述所有这些光滑的和尖峰的行进波！将其统一在一个理论框架下！

理论相容性

欧氏几何



非欧几何

光滑波浪



非光滑波浪



数学工具的重要性

尖峰孤立波

1993年首次提出，得到广泛研究，但至今都局限于浅水

同伦分析方法

1992年由本人提出，经过20年发展、完善，已成为一个有效的研究手段

2013年: 提出“统一波浪模型”
首次获得有限深水尖峰孤立波

真正新的理论或方法，终将给出一些全新的东西！



传统观点:

- 孤立波波速与波高密切相关
- 从波面到底部，波幅呈指数衰减
- 孤立波满足非线性的微分方程

质疑和挑战传统，
是科学家的天职！

全新的观点:

- 尖峰孤立波波速与波高无关！
- 从波面到底部，尖峰孤立波波幅几乎不衰减！
- 线性微分方程也可以给出孤立波！



局限性:

与传统光滑波浪理论一样，“统一波浪理论”是建立在理想流体**假设**之上。

但真实流体都具有粘性。因此，理论与现实之间一定存在某些**差异**。

后续研究:

1. **粘性**对有限水深尖峰孤立波**有何影响**？
2. 有限水深尖峰孤立波**如何形成**？其形成机理是什么？
3. 能否**实验验证**该尖峰孤立波之存在？



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请各位专家批评指正
谢谢！

