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# Catastrophic rupture of heterogeneous brittle materials under impact loading

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**Abstract.** The catastrophic failure of heterogeneous brittle materials under impact loading is not fully understood. To describe the catastrophic failure behavior of heterogeneous brittle materials under impact loading, an elasto-statistical-brittle (ESB) model is proposed in this paper. The ESB model characterizes the disordered inhomogeneity of material at mesoscopic scale with the statistical description of the shear strength of mesoscopic units. If the applied shear stress reaches the strength, the mesoscopic unit fails, which causes degradation in the shear modulus of the material. With a simplified ESB model, the failure wave in brittle material under uni-axial compression is analyzed. It is shown that the failure wave is a wave of strain or particle velocity resulted from the catastrophic fracture in an elastically stressed brittle media when the impact velocity reaches a critical value. In addition, the failure wave causes an increase in the rear surface velocity, which agrees well with experimental observations. The critical condition to generate failure wave and the speed of failure wave are also obtained.

## 1. INTRODUCTION

The catastrophic failure of heterogeneous brittle materials under impact loading is not fully understood, although many advances have been made in the characterization, visualization, and modeling of this complex process. Experiments suggest that the failure is resulted from the initiation, growth, and coalescence of microcracks [1], which is obviously controlled by the interactions of stress pulses with local material properties, microstructure, internal material interfaces, etc. Hence, the phenomenon is fundamentally different from those explained by conventional macroscopic descriptions. The heterogeneity at the mesoscopic scale, the induced uncertainty at macroscopic level, as well as the dynamic issue of heterogeneous and non-equilibrium shock processes present fundamental difficulties for the modeling and simulation of the process [2].

Micro-mechanical models have been developed to describe the catastrophic rupture process of heterogeneous brittle material under impact loading [3–6]. These models capture some of the underlying physical mechanisms during a brittle failure process, but either assume dilute pre-existing crack distributions with no interactions or model the micro-crack interaction by assuming a periodic or uniform spatial distribution of micro-cracks. In [7, 8], a model based on the failure threshold distribution of mesoscopic units was proposed and employed to the simulation of failure of rocks and cements. In this model, the damage evolution is governed by the tensile stress applied, and hence can only be applied to the simulation of extension fracture.

Motivated by the abovementioned facts, in this paper, an elasto-statistical-brittle (ESB) model is proposed. Similar to [7, 8], in the ESB model, the disordered inhomogeneity of material at mesoscopic scale is characterized with the statistical description of the shear strength of mesoscopic units. The non-equilibrium evolution of microdamage is controlled by the strength and the shear stress applied. With the ESB model, the stress-strain relationship in uni-axial strain state is obtained and a typical phenomenon of catastrophic failure of brittle media under impact loading, the failure wave in brittle material under uni-axial compression, is analyzed. The paper is organized as follows. Section 2 describes the ESB model. Section 3 offers a detailed analysis of failure wave with the model. Section 4 summarizes this investigation.

## 2. ELASTO-STATISTICAL-BRITTLE (ESB) MODEL

Suppose the medium be linear elastic but heterogeneous in brittle fracture, namely all mesoscopic units have the same linear elastic modulus but each mesoscopic unit has its own shear strength $\tau_c$. Hence, the heterogeneity of the mesoscopic units can be characterized by their shear strength. Generally, we assume that the shear strength $\tau_c$ follows a statistical distribution function $h(\tau_c)$. If the applied shear stress on the mesoscopic unit $\tau_{\text{meso}}$ reaches its shear strength, the unit fails. Denote damage fraction of the macro-element as $D$, and damage $D$ can be expressed by

$$D = \int_0^{\tau_{\text{meso}}} h(\tau_c)\, d\tau_c. \tag{1}$$

Assume the distribution function $h(\tau_c)$ is of Weibull type,

$$h(\tau_c) = h(\tau_c) = \frac{m}{\eta}\left(\frac{\tau_c}{\eta}\right)^{m-1} \exp\left[-\left(\frac{\tau_c}{\eta}\right)^{m}\right], \tag{2}$$

where $m > 0$ is the shape parameter characterizing the degree of heterogeneity, and $\eta > 0$ is the scale parameter proportional to the mean strength of mesoscopic units. Then, the macroscopic damage is

$$D = 1 - \exp\left[-\left(\frac{\tau_{\text{meso}}}{\eta}\right)^{m}\right]. \tag{3}$$

Provided small damage and mean field approximation, $\tau_{\text{meso}} = G_0\gamma$

$$D = 1 - \exp\left[-\left(\frac{G_0\gamma}{\eta}\right)^{m}\right]. \tag{4}$$

where $G_0$ is the shear modulus of intact material, $\gamma$ is the maximum shear strain of the macroscopic element.

We further assume that after brittle breaking, the mesoscopic units can still sustain elastic volumetric strain. Noticeably, this supposition can hold, only if the medium is quite heterogeneous and micro-fracture can be stopped by surrounding heterogeneous meso-elements, otherwise a macro-fracture will form instead. This assumption physically means that the macroscopic damage only causes a reduction in shear modulus, but no degradation in volumetric modulus of the material. That is,

$$G = G_0(1 - D) = G_0 \exp\left[-\left(\frac{G_0\gamma}{\eta}\right)^{m}\right], \tag{5}$$

$$K = K_0. \tag{6}$$

Therefore, we obtain the stress-strain relationship as

$$\sigma_{ij} = K_0\theta\delta_{ij} + 2G_0 \exp\left[-\left(\frac{G_0\gamma}{\eta}\right)^{m}\right]\left(\varepsilon_{ij} - \frac{1}{3}\theta\delta_{ij}\right). \tag{7}$$

In uni-axial strain state, $\theta = \gamma = \varepsilon_x$, formulas (7) can be simplified as

$$\begin{aligned}\sigma_x &= K_0\varepsilon_x + \frac{4}{3}G_0 \exp\left[-\left(\frac{G_0\varepsilon_x}{\eta}\right)^m\right]\varepsilon_x \\ \sigma_y &= \sigma_z = K_0\varepsilon_x - \frac{2}{3}G_0 \exp\left[-\left(\frac{G_0\varepsilon_x}{\eta}\right)^m\right]\varepsilon_x \\ \tau_{xy} &= \tau_{xz} = G_0 \exp\left[-\left(\frac{G_0\varepsilon_x}{\eta}\right)^m\right]\varepsilon_x, \quad \tau_{yz} = 0\end{aligned} \tag{8}$$

If we normalize the stress with $\eta$, strain with $\frac{\eta}{G_0}$, the expressions (8) becomes

$$\begin{aligned}\bar{\sigma}_x &= \frac{2(1+\nu)}{3(1-2\nu)}\bar{\varepsilon}_x + \frac{4}{3}\bar{\varepsilon}_x \exp\left(-\bar{\varepsilon}_x^m\right) \\ \bar{\sigma}_y &= \bar{\sigma}_z = \frac{2(1+\nu)}{3(1-2\nu)}\bar{\varepsilon}_x - \frac{2}{3}\bar{\varepsilon}_x \exp\left(-\bar{\varepsilon}_x^m\right) \\ \bar{\tau}_{xy} &= \bar{\tau}_{xz} = \bar{\varepsilon}_x \cdot \exp\left(-\bar{\varepsilon}_x^m\right), \quad \tau_{yz} = 0\end{aligned} \tag{9}$$

where $\nu$ is the Possion's ratio of the intact material. Figures 1 (a) and (b) plot the normalized stress-strain relationship with different parameters. It is noticeable that there exists a saddle point in the $\sigma_x - \varepsilon_x$ curve with parameters $\nu = 0.3$, $m = 10$. Actually, the saddle-point implies a catastrophic transition happening. In addition, we can derive that the saddle point exists when $m \geq m_0 = \left[LambertW\left(\frac{2(1-2\nu)}{(1+\nu)e}\right)\right]^{-1}$. For $\nu = 0.3$, $m_0 = 5.329$; $\nu = 0.25$, $m_0 = 4.29$.

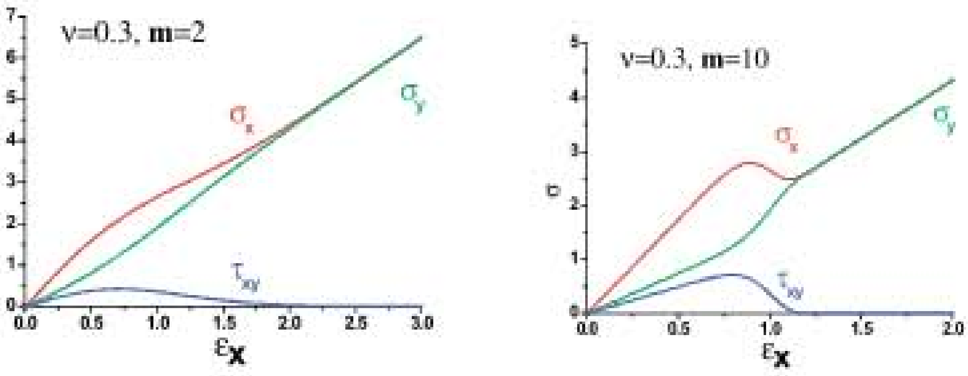


**Figure 1.** Normalized stress-strain curves in uniaxial state. (a) $\nu = 0.3$, $m = 2$; (b) $\nu = 0.3$, $m = 10$.

## 3. FAILURE WAVE ANALYSIS WITH A SIMPLIFIED ESB MODEL

Since 1991, failure waves have been observed in a wide range of brittle materials under impact loading [9–11]. To sum up, the main features of failure waves are: failure waves initiate at the impact surface and propagate into the sample plate behind the compressive longitudinal wave; the speed of failure wave increases with increasing impact loading; behind the failure wave the transverse stress increases, shear stress decreases and the tensile strength drops to essentially zero. A number of papers have addressed the origins of failure waves [12–15]. However, the theoretical interpretation of the formation of failure waves remains open. In this section, we will illustrate the failure wave with a simplified ESB model.

As Figure 1 (b) shows, if $m \geq m_0 = \left[LambertW\left(\frac{2(1-2\nu)}{(1+\nu)e}\right)\right]^{-1}$, a catastrophic transition happens, which corresponds to a saddle point in the normalized $\sigma_x - \varepsilon_x$ curve of the material. And then, we can sketch the $\sigma_x - \varepsilon_x$ curve in Figure 1 (b) as Figure 2. In Figure 2, the first peak value of stress is denoted by $\sigma_E$, the corresponding strain by $\varepsilon_E$ and $\varepsilon_K$. It is worth noticing that due to the catastrophic rupture, the state will directly jump from $(\sigma_E, \varepsilon_E)$ to $(\sigma_E, \varepsilon_K)$ if $\sigma_x$ reaches $\sigma_E$.

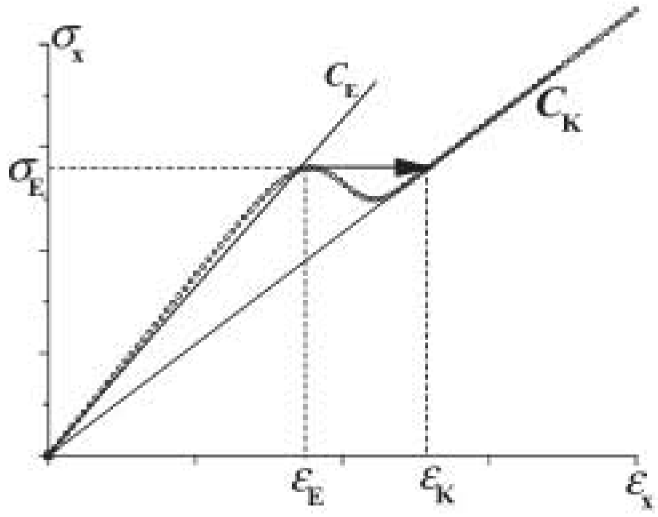


**Figure 2.** A sketch of simplified ESB model.

Let's apply the simplified ESB model to analyze a normal plate impact. Figure 3 is the *X-T* diagram for waves in normal plate impact. Note that if the velocity of the flying plate is high enough, there exist an elastic precursor and a failure wave, marked by the solid and the dashed line respectively in the diagram. $C_{\mathrm{E}}$ and $C_{\mathrm{fw}}$ are the speed of elastic precursor and failure wave in the target plate, respectively. $C_{\mathrm{Ef}}$ is the speed of elastic wave in the flying target. Obviously, $\sigma_{\mathrm{E}} = \rho_0 C_{\mathrm{E}}^2 \varepsilon_{\mathrm{E}}$ and $\sigma_{\mathrm{E}} = \rho_f C_{\mathrm{Ef}}^2 \varepsilon_{f\mathrm{E}}$. In addition, we denote the particle velocity, strain, and stress in region $i$ by $v_i, \varepsilon_i, \sigma_i$ , which can be determined by the method of characteristics.

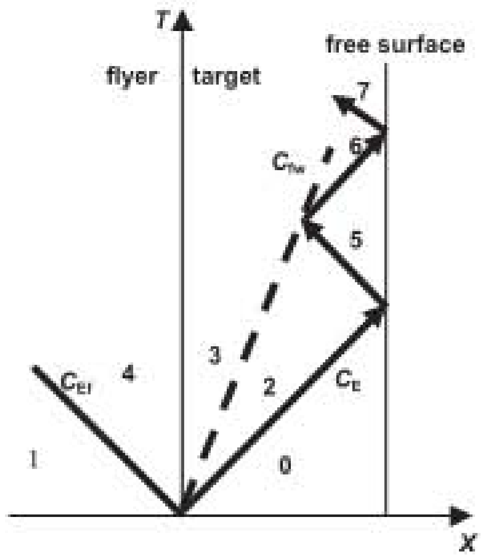


**Figure 3.** *X-T* diagram for waves. The solid line and dashed line denote elastic wave and failure wave, respectively.

The state in region 0 and region 1 are the initial states in the target and flying plate, respectively. That is, in region 0, $v_0 = 0, \varepsilon_0 = 0,\ \sigma_0 = 0$; in region 1, $v_1 = v_f,\ \varepsilon_1 = 0,\ \sigma_1 = 0$.

In region 2, the stress and strain are $\sigma_E$ and $\varepsilon_E$, respectively. The particle velocity $v_2$ can be obtained from $C_\sigma = \frac{1}{\rho_0}\left(\frac{\Delta\sigma}{\Delta v}\right)_h = \frac{1}{\rho_0}\left(\frac{\sigma_2-\sigma_1}{v_2-v_1}\right) = C_E$, and $v_2 = \frac{\sigma_E}{\rho_0 C_E}$.

In region 3, the stress and strain are $\sigma_E$ and $\varepsilon_K$, respectively. Since the state 3 differs from state 2 only in the strain and particle velocity, the failure wave between state 2 and 3 is a wave of strain or particle velocity. And hence, the particle velocity $v_3$ can be calculated from:

$$C_v = \left(\frac{\Delta v}{\Delta \varepsilon}\right)_h = \left(\frac{v_3 - v_2}{\varepsilon_3 - \varepsilon_2}\right) = C_{fw} \tag{10}$$

In region 4, the stress and strain are $\sigma_E$ and $\varepsilon_{fE}$, respectively. The particle velocity $v_4$ can be calculated from:

$$C_\sigma = -\frac{1}{\rho_f}\left(\frac{\Delta\sigma}{\Delta v}\right)_h = -\frac{1}{\rho_f}\left(\frac{\sigma_4 - \sigma_1}{v_4 - v_1}\right) = C_{fE} \tag{11}$$

Considering that $\sigma_4 = v_3$, $v_4 = v_3$, Equations (10) and (11) should be solved simultaneously. Therefore, we obtain,

$$v_3 = v_4 = v_f - \frac{\sigma_E}{\rho_f C_{fE}} \tag{12}$$

$$C_{fw} = \frac{v_f - C_{fE}\varepsilon_{fE} - C_E\varepsilon_E}{\varepsilon_K - \varepsilon_E} \tag{13}$$

Equation (13) gives the velocity of failure wave. Obviously, if only $v_f > C_{fE}\varepsilon_{fE} + C_E\varepsilon_E$, $C_{fw} > 0$, which means failure wave appears. In addition, Equation (13) indicates that the velocity of failure wave increases with the velocity of flying plate, which agrees well with experimental results.

Similarly, we obtain the state in region 5, 6, 7 as $\sigma_5 = \varepsilon_5 = 0$, $v_5 = \frac{2\sigma_E}{\rho_0 C_E}$; $\varepsilon_6 = \varepsilon_E + \frac{2C_E\varepsilon_E(\varepsilon_K - \varepsilon_E)}{v_f - C_{fE}\varepsilon_{fE} - C_E\varepsilon_K}$, $\sigma_6 = \sigma_E + \frac{2C_E\sigma_E(\varepsilon_K - \varepsilon_E)}{v_f - C_{fE}\varepsilon_{fE} - C_E\varepsilon_K}$, $v_6 = \frac{3\sigma_E}{\rho_0 C_E} + \frac{2C_E^2\varepsilon_E(\varepsilon_K - \varepsilon_E)}{v_f - C_{fE}\varepsilon_{fE} - C_E\varepsilon_K}$; and $\sigma_7 = \varepsilon_7 = 0$, $v_7 = \frac{4\sigma_E}{\rho_0 C_E} + \frac{4\sigma_E(\varepsilon_K - \varepsilon_E)}{\rho_0\left(v_f - C_{fE}\varepsilon_{fE} - C_E\varepsilon_K\right)}$. Since $\varepsilon_K > \varepsilon_E$ and $v_f > C_{fE}\varepsilon_{fE} + C_E\varepsilon_K$, $v_7 > v_5$, which corresponds to a reload signal in the rear surface velocity traces observed in experiments.

## 4. SUMMARY

To describe the catastrophic failure behavior of heterogeneous brittle materials under impact loading, an elasto-statistical-brittle (ESB) model is proposed in this paper. The ESB model characterizes the disordered inhomogeneity of material at mesoscopic scale with the statistical description of the shear strength of mesoscopic unit. If the applied shear stress reaches the strength, the mesoscopic unit fails and causes degradation in the shear modulus of the material. With the ESB model, the stress-strain relationship in uni-axial strain state is obtained. It is found that if $m \geq m_0 = \left[LambertW\left(\frac{2(1-2\nu)}{(1+\nu)e}\right)\right]^{-1}$, there exists a saddle point in the $\sigma_x - \varepsilon_x$ curve, which corresponds to a catastrophic rupture of the material.

With a simplified ESB model, the failure wave in brittle material under uni-axial compression is analyzed. It's shown that the failure wave is a wave of strain or particle velocity resulted from the catastrophic fracture in an elastically stressed brittle media when the impact velocity reaches a critical value. In addition, the failure wave appears when $v_f > C_{fE}\varepsilon_{fE} + C_E\varepsilon_E$ and its velocity increases with the velocity of flying plate. The result also proves that the failure wave will cause an increase of rear surface velocity, which corresponds to a reload signal in the rear surface velocity traces observed in experiments.

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