Three-Dimensional Linear Instability Analysis of Thermocapillary Return Flow on a Porous Plane

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A three-dimensional linear instability analysis of thermocapillary convection in a fluid-porous double layer system, imposed by a horizontal temperature gradient, is performed. The basic motion of fluid is the surface-tension-driven return flow, and the movement of fluid in the porous layer is governed by Darcy's law. The slippery effect of velocity at the fluid-porous interface has been taken into account, and the influence of this velocity slippage on the instability characteristic of the system is emphasized. The new behavior of the thermocapillary convection instability has been found and discussed through the figures of the spectrum.

For most kinds of Newtonian fluid, its surface tension is usually a monotonous decreasing function of temperature. Since the pioneering work of Pearson[1] on a pure liquid layer with a free, undeformed surface, and heated from the opposite boundary, we know that the surface tension gradient caused by temperature perturbation could lead to convective instabilities. If a temperature gradient parallel to the surface of a fluid layer is imposed into such a kind of system, thermocapillary flow can take place in the basic state. Smith and Davis[2,3] first studied this kind of flow and its instability problem in a single-liquid-layer system. They considered the bottom boundary as a non-slippery, adiabatic wall. The basic flows, such as a Couette flow with a vertical linear distribution of horizontal velocity, and a return flow with zero mass flux through any vertical plane were investigated. For the convection of instability, they found propagating hydrothermal waves and stationary longitudinal rolls. In the system of two-layered immiscible fluids, the thermocapillary flow and instabilities have been analyzed in our previous works.[4,5] The Rayleigh-Bénard instability[7] in the double layer system, which is composed of a fluid layer overlying a porous layer saturated with the same liquid, and which is heated, was firstly studied by Chen and Chen[6] in 1988. After them, Straughan[8] made a linear analysis of the Marangoni convection in a similar system but a free upper surface. The velocities of the basic state are zero in all of their works. In 2005, Chang[9,10] first introduced the plane Couette flow and the plane Poiseuille flow as the basic motion in this kind of system. Because the upper boundary needs to be a rigid plane, the convective instability in his works can only be driven by gravity.

In this Letter, we impose a horizontal temperature gradient into the system of a fluid layer overlying a porous medium, and simplify the system into a single fluid layer due to the consideration of no instability convection in the porous layer. The return flow is chosen as the basic motion and the three-dimensional linear instability is analyzed.
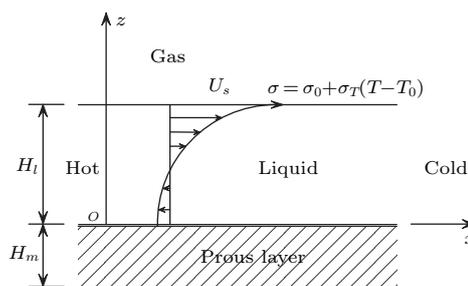

Fig. 1. The geometrical configuration.

Figure 1 is the schema of the system. Cartesian coordinates are used with the origin at the fluid-porous interface and the z axis is perpendicular to the interface and vertically upward. Three axes, i.e., x, y and z, organize a right-handed coordinate system. The upper surface is free without any deformation. On the rigid bottom wall there is zero heat flux. The surface tension at the upper surface is considered to be a linear function of temperature:

$$\sigma = \sigma_0 + \frac{\partial \sigma}{\partial T}(T - T_0). \qquad (1)$$

The subscript 0 denotes the ambient values. For convenience, we define $\gamma = -\partial\sigma/\partial T$ and $\bar{b} = -\partial T/\partial x$.

We define the dimensional variables with subscript l representing those in the fluid layer, and the subscript m for those in the porous layer. Therefore, without gravity, for the two layers, the continuity equa-

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tions are

$$\nabla \cdot \boldsymbol{v}_l = 0, \qquad (2)$$

$$\nabla \cdot \boldsymbol{v}_m = 0. \qquad (3)$$

The momentum equations are the Navier–Stokes equation[11] and Darcy's law,[12] respectively,

$$\rho_0 \Big(\frac{\partial \boldsymbol{v}_l}{\partial t} + \boldsymbol{v}_l \cdot \nabla \boldsymbol{v}_l \Big) = -\nabla p_l + \mu_l \nabla^2 \boldsymbol{v}_l, \qquad (4)$$

$$\frac{\rho_0}{\phi} \frac{\partial \boldsymbol{v}_m}{\partial t} = -\nabla p_m - \frac{\mu_l}{K} \boldsymbol{v}_m. \qquad (5)$$

The energy equations are

$$(\rho_0 c_0) \frac{\partial T_l}{\partial t} + (\rho_0 c_0) \boldsymbol{v}_l \cdot \nabla T_l = k_l \nabla^2 T_l, \qquad (6)$$

$$(\rho c)_m \frac{\partial T_m}{\partial t} + (\rho_0 c_0) \boldsymbol{v}_m \cdot \nabla T_m = k_m \nabla^2 T_m, \qquad (7)$$

wherein $\boldsymbol{v}$ is the velocity vector in the flow field, ρ the density of the fluid, c the specific heat of the fluid at constant pressure, μ_l the dynamic viscosity, $\nu_l = \mu_l/\rho$ is the kinetic viscosity, $k_{l,m}$ is the thermal conductivity, and $\kappa_{l,m} = k_{l,m}/(\rho_0 c_0)$ is the thermal diffusivity. For any dimensional physical property, one has $(\ )_m = (\ )_l \phi + (\ )_s(1-\phi)$, and ϕ is the porosity, the subscript s is for a solid matrix in the porous layer.[14]

The velocities in basic state are considered as $\boldsymbol{v}_{l,m} = (u_{l,m}, 0, 0)$, wherein $u_{l,m}$ are single-value functions of the vertical coordinate z. The corresponding boundary conditions are, at the top free surface $z = H_l$:

$$\mu_l \frac{du_l}{dz} = \gamma \bar{b}. \qquad (8)$$

At the fluid-porous interface $z = 0$, it is the Beaver–Joseph condition[13] with dimensionless experiential parameter α:

$$\frac{du_l}{dz} = \frac{\alpha}{\sqrt{K}}(u_l - u_m), \qquad (9)$$

where K is the permeability of the porous layer. Thus, the dimensional velocities of the basic state are as follows:

$$u_l(z) = \frac{1}{2\mu_l} \frac{\partial p_l}{\partial x} z^2 - \Big(\frac{\gamma}{\mu_l} \frac{\partial T_l}{\partial x} + \frac{H_l}{\mu_l} \frac{\partial p_l}{\partial x} \Big) z - \frac{K}{\mu_l} \frac{\partial p_m}{\partial x}$$
$$- \frac{\sqrt{K}}{\beta} \Big(\frac{\gamma}{\mu_l} \frac{\partial T_l}{\partial x} + \frac{H_l}{\mu_l} \frac{\partial p_l}{\partial x} \Big), \qquad (10)$$

$$u_m(z) = -\frac{K}{\mu_l} \frac{\partial p_m}{\partial x}. \qquad (11)$$

Through a comparison between Eqs. (10) and (11), we know the fluid velocity in the porous layer is much less than that in the fluid layer. Therefore, the flow in the porous layer can be neglected. A similar simplification has been employed in the works of Pascal[15] and Sadiq,[16] and is also used in the present study in both the basic state and perturbation state. To obtain the dimensionless form of equations, we find separate scales for the two layers respectively. For the fluid layer, we choose the characteristic length to be H_l, time to be H_l^2/κ_l, velocity to be ν_l/H_l, pressure to be $\rho_0 \nu_l^2 H_l^2$, and temperature to be $\bar{b} H_l$. For the porous layer, we choose H_m, $H_m^2 \kappa_m$, ν_l/H_m, $\rho_0 \nu_l^2 H_m^2$ and $\bar{b} H_m$. In the dimensionless form, variables without subscript are for those in fluid layer, while those in the porous layer are the same as the dimensional form. Hence, the dimensionless form of the basic state is

$$u(z) = \frac{1}{2} \frac{\partial p}{\partial x} z^2 + \Big(\frac{Ma}{Pr} - \frac{\partial p}{\partial x} \Big) z + \beta \Big(\frac{Ma}{Pr} - \frac{\partial p}{\partial x} \Big), \ (12)$$

$$u_m(z_m) = 0. \qquad (13)$$

The corresponding temperature is

$$T(z) = Pr \Big[-\frac{1}{24} \frac{\partial p}{\partial x} z^4 - \frac{1}{6} \Big(\frac{Ma}{Pr} - \frac{\partial p}{\partial x} \Big) z^3$$
$$- \frac{\beta}{2} \Big(\frac{Ma}{Pr} - \frac{\partial p}{\partial x} \Big) z^2 - \Big(\frac{1}{8} + \frac{\beta}{2} \Big) \frac{\partial p}{\partial x}$$
$$+ \Big(\frac{1}{6} + \frac{\beta}{2} \Big) \frac{Ma}{Pr} \Big], \qquad (14)$$

$$T_m(z_m) = -h Pr \Big(\frac{1}{8} + \frac{\beta}{2} \Big) \frac{\partial p}{\partial x} + h Ma \Big(\frac{1}{6} + \frac{\beta}{2} \Big). \ (15)$$

The dimensionless parameters are defined as follows:

$$Ma = \frac{\gamma \bar{b} H_l^2}{\mu_l \kappa_l}, \ Pr = \frac{\nu_l}{\kappa_l}, \ \delta = \frac{\sqrt{K}}{H_m}, \ h = \frac{H_l}{H_m}, \ \beta = \frac{\delta}{\alpha h}.$$

For linear instability analysis, we introduce perturbations of velocities, pressure and temperature: $\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{v}'$, $p = \bar{p} + p'$, $T = \bar{T} + T'$, into Eqs. (2)–(7), where the variables with an overline represent the basic state. According to the normal mode technique, we can find the solutions in the form

$$(u', v', w', T') = (U(z), V(z), W(z), \Theta(z))$$
$$\cdot \exp[\lambda t + i(ax + by)],$$

$$T'_m = \Theta_m(z_m) \exp[\lambda_m t_m + i(a_m x_m + b_m y_m)].$$

The amplitudes $U(z), V(z), W(z), \Theta(z), \Theta_m(z_m)$ describe the variation with respect to z; a, a_m and b, b_m are the dimensionless wavenumbers in the x- and y-directions, respectively, and $\{a, b\} = h\{a_m, b_m\}$, while $\lambda = Xh^2 \lambda_m$ are the growth rate factors in the fluid and porous layer. Then the linearized small disturbance equations in normal-mode form are

$$\frac{1}{Pr} \lambda U = [D^2 - (a^2 + b^2)]U - ia\bar{u}U - \frac{d\bar{u}}{dz}W - iaP, \ (16)$$

$$\frac{1}{Pr} \lambda V = [D^2 - (a^2 + b^2)]V - ia\bar{u}V - ibP, \qquad (17)$$

$$\frac{1}{Pr} \lambda W = [D^2 - (a^2 + b^2)]W - ia\bar{u}W - DP, \quad (18)$$

$$0 = iaU + ibV + DW, \qquad (19)$$

$$\frac{1}{Pr}\lambda\Theta = -\frac{\partial \overline{T}}{\partial x}U - \frac{\partial \overline{T}}{\partial z}W, + \frac{1}{Pr}[D^2 - (a^2 + b^2)]\Theta - ia\bar{u}\Theta,$$
(20)

$$G_m\lambda_m\Theta_m = [D_m^2 - (a_m^2 + b_m^2)]\Theta_m.$$
(21)

Boundary conditions in normal modes are as follows: for $z_m = -1$,

$$D_m\Theta_m = 0;$$
(22)

for $z = 0$,

$$W = 0, \quad DU = \frac{1}{\beta}U, \quad DV = \frac{1}{\beta}V,$$

$$h\Theta = \Theta_m, \quad D\Theta = XD_m\Theta_m;$$
(23)

for $z = 1$,

$$W = 0, \quad D\Theta + Bi\Theta = 0,$$

$$DU = -ia\frac{Ma}{Pr}\Theta, \quad DV = -ib\frac{Ma}{Pr}\Theta,$$
(24)

where $D = d/dz$, $D_m = d/dz_m$ and alphabetic subscripts denote partial differentiations. In order to study the three-dimensional perturbation, we define $k = \sqrt{a^2 + b^2}$, and $\theta = \arccos(a/k) = \arcsin(b/k)$. The direction of the perturbation wave can be denoted by θ. For the following discussion, we need to define the phase speed $c = -\lambda/ik$.[17] Every c corresponds to a λ.

The linear systems described above are discretized using the spectral method (Chebyshev-tau) and then are resolved as the general eigenvalue problem.[18] This method has been verified by our previous works.[19] The complex time growth rates λ, λ_m are computed in complex double precision. The computational solutions have also been verified in comparison with the work of Smith et al.[2]

From Fig. 2(a), we can see that, unlike the results of Chen and Chen,[6] all the marginal stability curves (neutral curves) are of a single mode with only

one least value (Ma_c) each. Obviously, it is impossible for the mode transition to take place. As the growth of β, the stability curves descend, i.e., the critical Marangoni number diminishes. This means that the larger the β is, the more conveniently the system destabilizes. Because β can be seen as a parameter of viscous restriction to the motion in the fluid layer from the fluid-porous interface, the less β is, the stronger this restriction will be, and the more easily the system remains stable.

In Fig. 2(b) we give the relation curves of oscillatory frequencies versus wavenumbers. We can find that the oscillatory frequencies to every wavenumber consist of two opposite values, i.e., the time growth rates along each neutral curve in Fig. 2(a) are conjugate. This kind of instability is in the form of hydrothermal waves, which were first pointed out by Smith et al.[2] The convectional vortex will be in the form of a traveling wave in either the positive or negative direction of the y-axis, or a standing wave superposed by these two 'mirror' waves. As the change of β, the oscillatory frequency will not vary evidently. This means that the slippage at the fluid-porous interface has almost no influence on the phase speed of the convectional vortex.

The neutral curves and their corresponding oscillatory frequencies of the perturbations in the x-direction are shown in Figs. 3(a) and 3(b). Similar to the results in Fig. 2(a), all the marginal curves are a single mode, and Ma_c diminishes as β grows. However, unlike Fig. 2(b), the oscillatory frequencies are single valued, and remain negative. According to the definition of the phase speed c above, we can know that the perturbation wave travels in the positive x-direction, the same as the surface motion in the basic state. Hence, we believe that it is the basic motion of fluid that drives the traveling direction of perturbation.
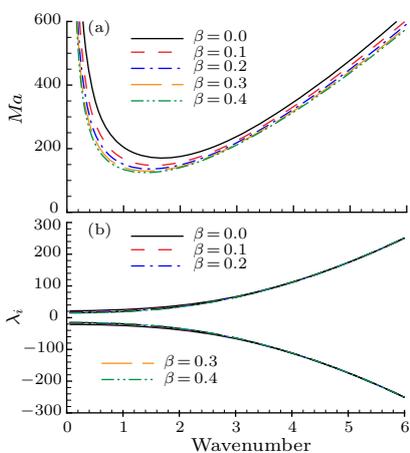

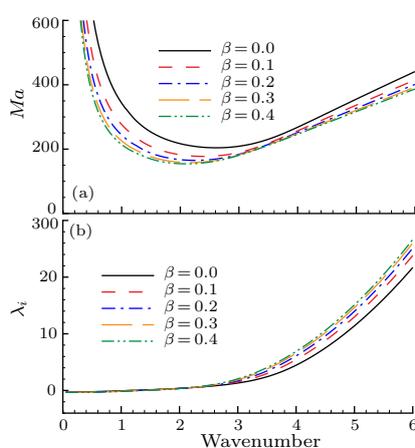**Fig. 2.** Neutral stability curves and corresponding oscillatory frequencies with different β for $a = 0$.

Fig. 3. Neutral stability curves and the corresponding oscillatory frequencies with different β for $b = 0$.
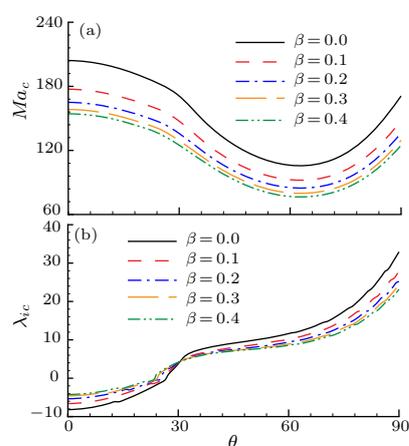

Fig. 4. Critical Marangoni numbers and corresponding critical oscillatory frequencies versus θ with different β.

In the three-dimensional problem, the Ma_c and λ_{ic} versus θ are shown respectively in Figs. 4(a) and 4(b). In Fig. 4(a), each curve has a minimum, and its corresponding θ, named θ_c, represents the most possible direction of the propagation when the instability takes place. We call it "the most dangerous direction". The variation of β will not influence θ_c evidently. In Fig. 4(b), for $\theta < 90°$, all the λ_{ic} are single valued. Only at $\theta = 90°$ does λ_{ic} have two values, one of which is the inverse of the other. The λ_{ic} varies monotonously from negative to positive during the growth of θ from $0°$ to $90°$. The value of θ_c corresponding to zero λ_{ic} is in the region $(24°, 30°)$. According to the results in Fig. 4(a), the three-dimensional convection of instability never has the standing wave mode.

From Fig. 4(a), we can find that no matter whether the perturbation is two-dimensional or three-dimensional, the system will be easier to destabilize with the increase of β. The slippage at the interface will not evidently affect the most dangerous direction. From Fig. 4(b), we can see that λ_{ic} is a monotonic increasing function of θ, but its growing speed will diminish as the increase of β. In other words, the slippage at the fluid-porous interface will decelerate the phase speed of the perturbation wave.
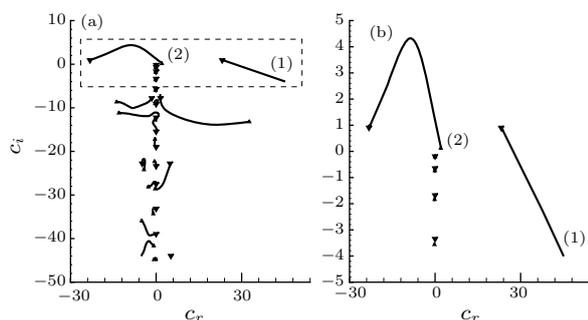

Fig. 5. The spectrum of eigenvalues at $k = 2$, $Ma = 220$ and $\beta = 0$ with the variation of θ from $0°$ to $90°$. (b) The magnification of the square region in (a). The symbol $\triangledown$ on every curve represents the values at $\theta = 90°$.

The trajectory of spectral points (c_r, c_i) during the variation of θ for fixing $k = 2$, $Ma = 220$ and $\beta = 0$ is shown in Fig. 5. From the definition of c, we know that c_i corresponds to the real part of λ. Therefore, the point with the largest c_i can influence the characteristic of instability. In Fig. 5, only the two points named No 1 and No 2 have this kind of possibility. When θ is less than $90°$, only No 2 has the largest c_i, so the perturbation wave can travel with only one possible phase speed c_r, which turns to negative from positive. At this time, the No 1 spectral point has the second largest c_i, so its character is "overlapped" by No 1. When θ arrives at $90°$, both No 1 and No 2 can reach the same value of c_i, and their c_r are inverse to each other. Hence, both of them can be "expressed", and this is the reason why the hydrothermal wave can take place. For the case where $\beta \neq 0$, the spectrum

of eigenvalues is similar to Fig. 5. The variation of β will not have an evident effect on the process of mode transition during the growth of θ. Those results are left out in this Letter.

In addition, in Fig. 5, $Ma = 220$ has already surpassed its critical value Ma_c for $k = 2$. In principle, in the case of $Ma > Ma_c$, the nonlinear effect should be taken account, but it has exceeded the extension in this study. However, the variation of Ma will not change the relative positions of these spectral points in spectral figures obviously, for in our system it has already been linearized. If we show the spectrum of the case $Ma = Ma_c$, the trajectory of spectral points will be similar to those in Fig. 5. Only the value of c_i of each spectral point becomes less, No 1 and No 2 terminate on the horizontal line $c_i = 0$. Therefore, we believe that the results in Fig. 5 have the proper physical significance.

In summary, since the motion of the basic state is the return flow, the neutral curve of stability is of a single mode, the slippage at the fluid-porous interface will only obviously influence the critical Marangoni number which determines the instability of the system, but will not evidently change the phase speed of the perturbation wave or the most dangerous direction. For any fixed θ less than $90°$, the perturbation has the form of a traveling wave, with only one possible phase speed. This has the result that the convectional vortex is driven by the basic motion. Only when the perturbation is perpendicular to the direction of basic flow do both the opposite phase speeds become possible at the same time, and the hydrothermal wave can take place.

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