

PHYSICS

Rheological effect on thermocapillary flow of a liquid film jet painted on a moving boundary

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Abstract In the present paper, a liquid (or melt) film of relatively high temperature ejected from a vessel and painted on the moving solid film is analyzed by using the second-order fluid model of the non-Newtonian fluid. The thermocapillary flow driven by the temperature gradient on the free surface of a Newtonian liquid film was discussed before. The effect of rheological fluid on thermocapillary flow is considered in the present paper. The analysis is based on the approximations of lubrication theory and perturbation theory. The equation of liquid height and the process of thermal hydrodynamics of the non-Newtonian liquid film are obtained, and the case of weak effect of the rheological fluid is solved in detail.

Keywords: die swell, thermocapillary flow, non-Newtonian fluid, film jet.

In the previous papers^[1,2], the thermocapillary flow in addition to a Newtonian liquid film jet painted on a moving boundary was studied. The thermocapillary flow is induced by the gradient of surface tension caused by the heat transfer from the melt of relatively high temperature to the environmental gas of relatively low temperature. The temperature of the liquid or the melt, ejected from a nozzle of a manufacture vessel, is higher than the environmental gas temperature, and then, there is a strong heat exchange, especially in the region near the exit of vessel. The heat transfer from the liquid film to the environmental gas forms a temperature gradient and then the surface tension gradient on the free surface, and induces the thermocapillary flow, which will, of course, change the cross-section of the jet liquid. The thermocapillary effect may enlarge the cross-section, and then apply to the polymer processing. In fact, the viscous and elastic properties of the polymers are various and complicated, and some of them can be described by the Newtonian fluid, but most of them, especially the polymer of larger molecular chain, show rheological property. Therefore, it is necessary to study the effect of rheological liquid on the thermocapillary flow.

The processing of film and polymer requires an understanding of the parameters of hydrodynamic process, such as temperature, pressure, flow field and the diameter or height distribution (see, for example, refs. [3, 4]). The Swell or Die Swell effect on the change of cross-section in a polymer jet is often observed in the processing, and is explained usually by the rheology property

of the liquid medium. Tanner suggested a theory of Die-Swell, which assumes that the velocity has only one component along the jet^[5]. Because of the non-uniformity of the cross-section, the flow field in the Die-Swell theory should be at least two-dimensional. Most explanations of the Die-Swell effect due to the rheological fluid property are qualitative^[6,7]. It is believed that the difference in normal stresses may enlarge the cross-section of the melt jet.

It is known that, the thermocapillary flow can be induced in a thin liquid layer located on a solid wall with non-uniform temperature distribution, and the heat transfer from the liquid to the gas will result in non-uniformity of the liquid thickness. By using the approximation of the lubrication theory, an ordinary differential equation of liquid thickness and its solution for given boundary temperature were obtained for unsteady case in a thin liquid layer^[8]. A similar method was employed to discuss the steady cases where a smooth condition at the symmetric cross-section was improved^[9]. The problem of motionless and infinitely extending solid boundary was studied in refs. [8, 9]. Furthermore, the experiments show that the solutal capillary flow induced by the surfactant of the liquid may increase the thickness of the liquid layer^[10].

In the present paper, the change of cross-section due to both the thermocapillary flow in a two-dimensional and steady model is discussed for the jet liquid film of a non-Newtonian fluid on a moving solid boundary. In the case of weak non-Newtonian fluid, the zero's order solution reduces to the problem of the Newtonian fluid case. The results show that both the rheological effect and the thermocapillary flow can enlarge the cross-section of the liquid jet.

1 The model of non-Newtonian liquid film jet

A simplified model is proposed as shown in fig.1, where the liquid film is ejected from a nozzle of a melt or liquid vessel and then attached on a moving solid film of constant thickness h_s , and the Cartesian coordinate system (x, y, z) is adopted. The two-dimensional process (x, z) is assumed with $\partial/\partial y = 0$. The melt or liquid flows from the exit of the vessel for a long distance.

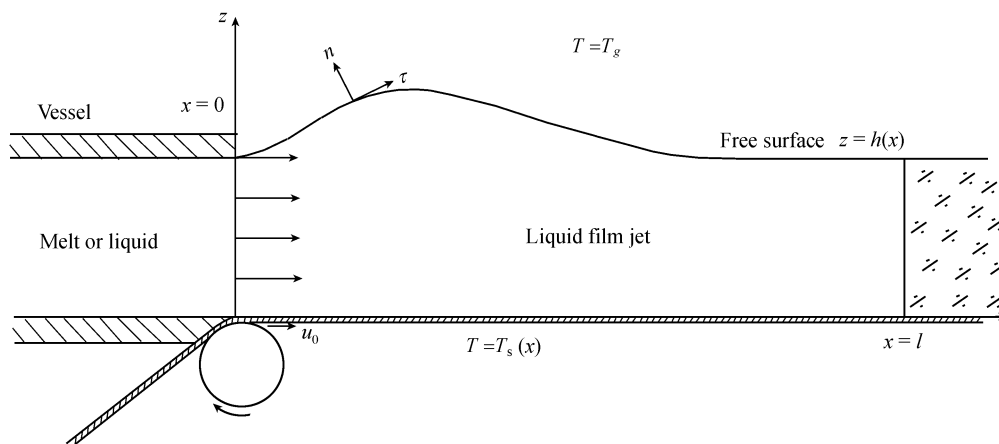


Fig. 1. Schematic diagram of the physical model of a non-Newtonian jet liquid film.

There is a small geometrical aspect ratio $\varepsilon = h_e/l \ll 1$, where h_e is the initial height of liquid layer at $x = 0$ and l is a typical distance in the longitudinal direction. The liquid temperature T_e at the exit is higher than the environmental gas temperature T_g and the temperature T_l at $x = l$. A moving solid film is touched smoothly with the lower boundary of the vessel, and moves with the same velocity u_s of the liquid at $z = 0$.

The liquid is considered as an incompressible fluid with constant kinematic viscosity ν and thermal diffusivity κ . The steady and two-dimensional conservation relationships of a non-Newtonian fluid may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (1.1)$$

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \left(\frac{\partial \pi_{xx}}{\partial x} + \frac{\partial \pi_{xz}}{\partial z} \right), \quad (1.2)$$

$$u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \left(\frac{\partial \pi_{xz}}{\partial x} + \frac{\partial \pi_{zz}}{\partial z} \right), \quad (1.3)$$

$$u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right), \quad (1.4)$$

where ρ , p and T are respectively the density, pressure and temperature of the liquid, $(u, 0, w)$ is velocity vector, and π_{xx} , π_{xz} , and π_{zz} are the components of a stress tensor $\boldsymbol{\pi}$.

The boundary conditions for the liquid layer can be written as^[1]

$$z=0: u=u_s, \quad w=0, \quad T=T_s(x), \quad (1.5)$$

$$z = h(x) : u \frac{dh}{dx} = w, \quad (1.6)$$

$$(-p + \pi_{xx})n_x^2 + 2\pi_{xz}n_x n_z + (-p + \pi_{zz})n_z^2 = \frac{2\sigma}{R_c}, \quad (1.7)$$

$$(-p + \pi_{xx})n_x \tau_x + \pi_{xz}(n_z \tau_x + n_x \tau_z) + (-p + \pi_{zz})n_z \tau_z = -\frac{|\sigma_T|}{\sqrt{1+h'(x)}} \left(\frac{\partial T}{\partial x} + h' \frac{\partial T}{\partial z} \right), \quad (1.8)$$

$$k \frac{\partial T}{\partial n} = -H(T - T_g), \quad (1.9)$$

where $h' = dh/dx$, T^* is a reference constant temperature, \mathbf{n} is the unit normal vector, k and H are respectively the thermal conductivity of liquid and the heat transfer coefficient of gas, and the radiation effect is omitted in (1.9). The unit normal vector \mathbf{n} and unit tangent vector $\boldsymbol{\tau}$ are respectively

$$(n_x, 0, n_z) = \frac{1}{\sqrt{1+h'^2}}(-h', 0, 1), \quad (\tau_x, 0, \tau_z) = \frac{1}{\sqrt{1+h'^2}}(1, 0, h').$$

And the curvature of the free surface is

$$\frac{1}{R_c} = \frac{h''}{2(1+h'^2)^{3/2}}.$$

Temperature $T_s(x)$ in condition (1.5) is the temperature at the boundary $z = 0$, and should be solved, in general, together with the Laplace equation and related boundary conditions for the solid temperature $T^s(x, z)$ in the film in region $-h_s \leq z \leq 0$. Obviously, a linear distribution of x is the solution of the problem, such as

$$T^s(x, z) = T_s(x) = T_e - (T_e - T_l)(x/l), \quad (1.10)$$

and T_e and T_l are respectively the temperatures at $x = 0$ and $x = l$.

To discuss the rheology fluid, the Coleman-Noll second order fluids based on the Rivlin-Ericksen stress tensor is introduced as follows (see, for example, (2.33) of ref. [11]):

$$\boldsymbol{\pi} = \mu \mathbf{A}_1 + \alpha_1^* \mathbf{A}_2 + \alpha_2^* \mathbf{A}_1^2, \quad (1.11)$$

where $\boldsymbol{\pi}$ is the stress tensor, \mathbf{A}_1 and \mathbf{A}_2 are Rivlin-Ericksen tensors; μ , α_1^* and α_2^* are viscoelastic coefficients, and there are $\alpha_1^* > 0$ and $\alpha_2^* < 0$. Coefficient μ reduces to the viscose coefficient of Newtonian fluid if $\alpha_1^* = \alpha_2^* = 0$. For the case of steady and two-dimensional problem

$$u = u(x, z), \quad v = 0, \quad w = w(x, z). \quad (1.12)$$

The components of stress tensor are written as

$$\pi_{xx} = 2\mu \frac{\partial u}{\partial x} + 4\alpha_1^* \left[\left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{4} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right] + 2\alpha_2^* \left[u \frac{\partial^2 u}{\partial x^2} + w \frac{\partial^2 u}{\partial x \partial z} + 2 \left(\frac{\partial u}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right], \quad (1.13)$$

$$\pi_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + 2\alpha_2^* \left[u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + w \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + 2 \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial z} \right) \right], \quad (1.14)$$

$$\pi_{zz} = 2\mu \frac{\partial w}{\partial z} + 4\alpha_1^* \left[\left(\frac{\partial w}{\partial z} \right)^2 + \frac{1}{4} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right] + 2\alpha_2^* \left[w \frac{\partial^2 w}{\partial z^2} + u \frac{\partial^2 w}{\partial x \partial z} + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \frac{\partial u}{\partial z} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right]. \quad (1.15)$$

The first normal stress difference $N_1 = -2\alpha_2^*$ is positive, and the second normal stress difference $N_2 = \alpha_1^* + 2\alpha_2^*$ is small. Substituting non-Newtonian relationship (1.13)—(1.15), basic equations (1.1)—(1.4) can be solved under the boundary conditions.

2 Lubrication approximation

Non-dimensional quantities and parameters may be introduced based on the lubrication theory as follows^[5]:

$$\begin{aligned} \xi = \frac{x}{l}, \quad \zeta = \frac{z}{h_e}, \quad \eta = \frac{h}{h_e}, \quad \varepsilon = \frac{h_e}{l}, \quad \alpha_1 = \frac{\alpha_1^* \nu_*}{\mu l}, \quad \alpha_2 = \frac{\alpha_2^* \nu_*}{\mu l}, \\ U = \frac{u}{\nu_*}, \quad \varepsilon W = \frac{w}{\nu_*}, \quad \varepsilon^2 P = \frac{p l}{\mu \nu_*}, \quad \Theta = \frac{T}{T_*}, \end{aligned} \quad (2.1)$$

where the typical temperature T_* and the typical velocity v_* are adopted respectively as

$$T_* = T_s(0), \quad v_* = -\varepsilon\sigma'T_*/\mu. \quad (2.2)$$

The basic feature of the lubrication problem is that there are two typical scales of different orders of magnitudes; that is, one of the typical scales l is much larger than the other h_e , and then other quantities have different orders of magnitude. The non-dimensional parameter in this case are the Reynolds number Re and Peclet number Pe

$$Re = \frac{v_*l}{\nu}, \quad Pe = \frac{v_*l}{\kappa}. \quad (2.3)$$

The Prandtl number and Marangoni number may be given as $Re = Pr Ma$ and $Ma = -\sigma_T'T_*/\kappa\nu = \varepsilon Pe$. Non-dimensional equations can then be written as

$$\frac{\partial U}{\partial \xi} + \frac{\partial W}{\partial \zeta} = 0, \quad (2.4)$$

$$\begin{aligned} Re\varepsilon^2 \left(U \frac{\partial U}{\partial \xi} + W \frac{\partial U}{\partial \zeta} \right) &= -\frac{\partial P}{\partial \xi} + \frac{\partial^2 U}{\partial \zeta^2} + \alpha_1 \frac{\partial}{\partial \xi} \left(\frac{\partial U}{\partial \zeta} \right)^2 + \varepsilon^2 \alpha_1 \left(2 \frac{\partial^3 U}{\partial \xi^3} + \frac{\partial^3 W}{\partial \xi \partial \zeta^2} \right) \\ &+ 4\varepsilon^2 \alpha_1 \frac{\partial}{\partial \xi} \left[\left(\frac{\partial U}{\partial \xi} \right)^2 + \frac{1}{2} \frac{\partial U}{\partial \zeta} \frac{\partial W}{\partial \xi} + \frac{\varepsilon^2}{4} \left(\frac{\partial W}{\partial \xi} \right)^2 \right] + \alpha_2 \frac{\partial}{\partial \zeta} \left[\left(U \frac{\partial}{\partial \xi} + W \frac{\partial}{\partial \zeta} \right) \frac{\partial U}{\partial \zeta} + 2 \frac{\partial U}{\partial \xi} \frac{\partial U}{\partial \zeta} \right] \\ &+ \varepsilon^2 \alpha_2 \left\{ 2 \frac{\partial}{\partial \xi} \left[U \frac{\partial^2 U}{\partial \xi^2} + W \frac{\partial U}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial U}{\partial \xi} \right)^2 + \frac{\partial W}{\partial \xi} \left(\frac{\partial U}{\partial \zeta} + \varepsilon^2 \frac{\partial W}{\partial \zeta} \right) \right] \right. \\ &\left. + \frac{\partial}{\partial \zeta} \left[2 \frac{\partial W}{\partial \xi} \frac{\partial W}{\partial \zeta} + \left(U \frac{\partial}{\partial \xi} + W \frac{\partial}{\partial \zeta} \right) \frac{\partial W}{\partial \xi} \right] \right\}, \quad (2.5) \end{aligned}$$

$$\begin{aligned} Re\varepsilon^4 \left(U \frac{\partial W}{\partial \xi} + W \frac{\partial W}{\partial \zeta} \right) &= -\frac{\partial P}{\partial \zeta} + (\alpha_1 + 2\alpha_2) \frac{\partial}{\partial \zeta} \left(\frac{\partial U}{\partial \zeta} \right)^2 + \varepsilon^2 \left[\frac{\partial}{\partial \xi} \left(\frac{\partial U}{\partial \zeta} + \varepsilon^2 \frac{\partial W}{\partial \zeta} \right) + 2 \frac{\partial^2 W}{\partial \zeta^2} \right] \\ &+ 4\varepsilon^2 \alpha_1 \frac{\partial}{\partial \zeta} \left[\left(\frac{\partial W}{\partial \zeta} \right)^2 + \frac{1}{2} \frac{\partial U}{\partial \zeta} \frac{\partial W}{\partial \xi} + \frac{\varepsilon^2}{4} \left(\frac{\partial W}{\partial \xi} \right)^2 \right] + \varepsilon^2 \alpha_2 \left\{ \frac{\partial}{\partial \xi} \left[\left(U \frac{\partial}{\partial \xi} + W \frac{\partial}{\partial \zeta} \right) \left(\frac{\partial U}{\partial \zeta} + \varepsilon^2 \frac{\partial W}{\partial \xi} \right) \right. \right. \\ &\left. \left. + 2 \left(\frac{\partial U}{\partial \xi} \frac{\partial U}{\partial \zeta} + \varepsilon^2 \frac{\partial W}{\partial \xi} \frac{\partial W}{\partial \zeta} \right) \right] + 2 \frac{\partial}{\partial \zeta} \left[W \frac{\partial^2 W}{\partial \zeta^2} + U \frac{\partial^2 W}{\partial \xi \partial \zeta} + 2 \left(\frac{\partial W}{\partial \zeta} \right)^2 + \frac{\partial U}{\partial \zeta} \frac{\partial W}{\partial \xi} \right] \right\}, \quad (2.6) \end{aligned}$$

$$Pe\varepsilon^2 \left(U \frac{\partial \Theta}{\partial \xi} + W \frac{\partial \Theta}{\partial \zeta} \right) = \frac{\partial^2 \Theta}{\partial \zeta^2} + \varepsilon^2 \frac{\partial^2 \Theta}{\partial \xi^2}; \quad (2.7)$$

and, similar to the those in ref. [1], the boundary conditions are given as

$$\zeta = 0: U = U_s(\text{const}), \quad W = 0, \quad \Theta = \Theta_s(\xi); \quad (2.8)$$

$$\zeta = \eta(\xi): W(\xi, \eta) = \eta'U(\xi, \eta), \quad (2.9)$$

$$\begin{aligned} & \frac{\partial U}{\partial \zeta} + \varepsilon^2 \left[\left(\frac{\partial W}{\partial \xi} - \eta^2 \frac{\partial U}{\partial \zeta} \right) + 2\eta \left(\frac{\partial W}{\partial \zeta} - \frac{\partial U}{\partial \xi} \right) - \varepsilon^2 \eta \frac{\partial W}{\partial \xi} \right] \\ & = \sqrt{1 + \varepsilon^2 \eta^2} \left(\frac{\partial \Theta}{\partial \xi} + \eta \frac{\partial \Theta}{\partial \zeta} \right) + \alpha_2 \left[2 \left(\frac{\partial U}{\partial \zeta} \right)^2 \eta' + O(\varepsilon) \right], \end{aligned} \quad (2.10)$$

$$\begin{aligned} P & = -\frac{\varepsilon^2}{C} \frac{\eta}{(1 + \varepsilon^2 \eta^2)^{3/2}} + \frac{2\varepsilon^2}{1 + \varepsilon^2 \eta^2} \left[\left(\frac{\partial W}{\partial \zeta} - \eta \frac{\partial U}{\partial \zeta} \right) + \varepsilon^2 \eta \left(-\frac{\partial W}{\partial \xi} + \eta \frac{\partial U}{\partial \xi} \right) \right] \\ & + (\alpha_1 + 2\alpha_2) \left(\frac{\partial U}{\partial \zeta} \right)^2 + O(\varepsilon^2), \end{aligned} \quad (2.11)$$

$$\frac{\partial \Theta}{\partial \zeta} - \varepsilon^2 \eta \frac{\partial \Theta}{\partial \xi} = -Bi(\Theta - \Theta_g)(1 + \varepsilon^2 \eta^2)^{1/2}, \quad (2.12)$$

where the Capillary number $C = -\sigma^* \tau T^* / \sigma_0$ is usually much smaller than unity, and the non-dimensional Boit number is defined by $Bi = Hh_0/k$.

3 Perturbation method

The non-dimensional equations and boundary conditions show the relationships of the orders of magnitude, and the perturbation method may be applied by the expansion based on the small parameter ε . By the order of magnitude analysis (OMA), it is required that

$$Re = O(1), Pe = O(1), Bi = O(1), C = \varepsilon^2/\alpha = O(\varepsilon^2), \quad (3.1)$$

where the coefficient of surface tension α is a constant, and the reological coefficients α_1 and α_2 are considered as $O(1)$ in this section. It is noted that definitions of the Reynolds number and the Peclet number in (3.3) are $1/\varepsilon$ that of the usual definitions, because the typical velocity v^* is $1/\varepsilon$ that of the usual typical thermocapillary velocity. The quantities may be expanded as follows:

$$U = \sum_{n=0}^{\infty} \varepsilon^n U_n, \quad W = \sum_{n=0}^{\infty} \varepsilon^n W_n, \quad \Theta = \sum_{n=0}^{\infty} \varepsilon^n \Theta_n, \quad \eta = \sum_{n=0}^{\infty} \varepsilon^n \eta_n. \quad (3.2)$$

Substituting relation (3.2) into the equations and boundary conditions, the problem may be solved order by order.

The zero's order relationships may be written as

$$\frac{\partial U_0}{\partial \xi} + \frac{\partial W_0}{\partial \zeta} = 0, \quad (3.3)$$

$$\frac{\partial P_0}{\partial \xi} = \frac{\partial^2 U_0}{\partial \zeta^2} + 2\alpha_1 \frac{\partial U_0}{\partial \xi} \frac{\partial^2 U_0}{\partial \xi \partial \zeta} + 2\alpha_2 \frac{\partial}{\partial \zeta} \left[\left(U_0 \frac{\partial}{\partial \xi} + W_0 \frac{\partial}{\partial \zeta} \right) \frac{\partial U_0}{\partial \zeta} + 2 \frac{\partial U_0}{\partial \xi} \frac{\partial U_0}{\partial \zeta} \right], \quad (3.4)$$

$$\frac{\partial P_0}{\partial \zeta} = (\alpha_1 + 2\alpha_2) \frac{\partial}{\partial \zeta} \left(\frac{\partial U_0}{\partial \zeta} \right)^2, \quad (3.5)$$

$$\frac{\partial^2 \Theta_0}{\partial \zeta^2} = 0, \quad (3.6)$$

and the boundary conditions are

$$\zeta = 0 : U_0 = U_s, \quad W_0 = 0, \quad \Theta_0 = \Theta_s(\xi), \quad (3.7)$$

$$\zeta = \eta_0 : W_0 = U_0 \eta_0', \quad (3.8)$$

$$\frac{\partial U_0}{\partial \zeta} = -\frac{\partial \Theta_0}{\partial \xi} - \frac{d\eta_0}{d\xi} \frac{\partial \Theta_0}{\partial \zeta} + 2\alpha_2 \left(\frac{\partial U(\xi, \eta_0)}{\partial \zeta} \right)^2 \frac{d\eta_0}{d\xi}, \quad (3.9)$$

$$P_0 = -\alpha \frac{d^2 \eta_0}{d\xi^2} + (\alpha_1 + 2\alpha_2) \left(\frac{\partial U(\xi, \eta_0)}{\partial \zeta} \right)^2, \quad (3.10)$$

$$\frac{\partial \Theta_0}{\partial \zeta} = -Bi(\Theta_0 - \Theta_g). \quad (3.11)$$

Basic equations (3.3) and (3.6) are the same as in ref. [1]. However, there are additional terms related to the rheological effect on the right-hand side of (3.4) and (3.5). Note that Θ_0 is decoupled and can be found by solving (3.6) to satisfy boundary conditions (3.7) and (3.11).

Eq. (3.6) means that the temperature is a linear function of ζ , and may be obtained as follows by using boundary conditions (3.7) and (3.11):

$$\Theta_0(\xi, \zeta) = \Theta_s(\xi) + f_0(\xi) \zeta, \quad f_0(\xi) = -\frac{Bi(\Theta_s - \Theta_g)}{1 + Bi\eta_0}. \quad (3.12)$$

Relationship $f_0(\xi)$ shows that the heat transfers from the solid of temperature Θ_s to the free surface of temperature $\Theta^{(0)}(\xi, \eta^{(0)})$, and then to the gas of temperature Θ_g .

Integrating eq. (3.5), we have

$$P_0 = (\alpha_1 + 2\alpha_2) \left(\frac{\partial U_0}{\partial \zeta} \right)^2 + P^*(\zeta), \quad (3.13)$$

where the integration function $P^*(\zeta)$ can be determined by boundary condition (3.10). Substituting relationship (3.13) into eq. (3.4), we have

$$\begin{aligned} & \frac{\partial^2 U_0}{\partial \zeta^2} + \alpha_2 \left[\frac{\partial U_0}{\partial \xi} \frac{\partial^2 U_0}{\partial \zeta^2} - \frac{\partial U_0}{\partial \zeta} \frac{\partial^2 U_0}{\partial \xi \partial \zeta} \right. \\ & \left. + \left(U_0 \frac{\partial}{\partial \xi} + W_0 \frac{\partial}{\partial \zeta} \right) \frac{\partial^2 U_0}{\partial \zeta^2} \right] = \frac{dP^*(\xi)}{d\xi}. \end{aligned} \quad (3.14)$$

Eq. (3.14) shows that the fluid field depends only on the elastic coefficient α_2 , and this conclusion agrees with the Tanner's theory^[12]. The solution of non-Newtonian fluid field can be solved by eqs. (3.3) and (3.14) under the boundary conditions for zero's order.

4 Weak non-Newtonian fluid

To simplify the problem, a weak non-Newtonian fluid approximation is discussed in detail in this section as the first step, and the coefficients α_1 and α_2 are written as

$$\alpha_1 = \varepsilon\beta_1, \quad \alpha_2 = \varepsilon\beta_2. \quad (4.1)$$

In addition to relationships (3.1), by substituting expansions (3.2) into the basic equations and boundary conditions, the relationships of zero's order reduce to that of Newtonian fluid, and are the same as those given in ref. [1]. The velocity fields of zero's order are

$$U_0(\xi, \zeta) = -\frac{\alpha}{2} \frac{d^3\eta_0}{d\xi^3} \zeta^2 + A(\xi)\zeta + U_s, \quad (4.2)$$

$$W_0(\xi, \zeta) = \frac{\alpha}{6} \frac{d^4\eta_0}{d\xi^4} \zeta^3 - \frac{1}{2} \frac{dA}{d\xi} \zeta^2, \quad (4.3)$$

where α is the coefficient of surface tension given by (3.1), and the function A is defined as

$$A(\xi) = \alpha\eta_0 \frac{d^3\eta_0}{d\xi^3} - \frac{d}{d\xi} [\Theta_s + f_0(\xi)\eta_0], \quad (4.4)$$

and $f_0(\xi)$ is given in (3.12).

For the weak non-Newtonian fluid approximation, the first order equations are

$$\frac{\partial U_1}{\partial \xi} + \frac{\partial W_1}{\partial \zeta} = 0, \quad (4.5)$$

$$\frac{\partial P_1}{\partial \xi} - \frac{\partial^2 U_1}{\partial \zeta^2} = \beta_1 \frac{\partial}{\partial \xi} \left(\frac{\partial U_0}{\partial \zeta} \right)^2 + \beta_2 \left[\left(U_0 \frac{\partial}{\partial \xi} + W_0 \frac{\partial}{\partial \zeta} \right) \frac{\partial U_0}{\partial \zeta} + 2 \frac{\partial U_0}{\partial \xi} \frac{\partial U_0}{\partial \zeta} \right], \quad (4.6)$$

$$\frac{\partial P_1}{\partial \zeta} = (\beta_1 + 2\beta_2) \frac{\partial}{\partial \zeta} \left(\frac{\partial U_0}{\partial \zeta} \right), \quad (4.7)$$

$$\frac{\partial^2 \Theta_1}{\partial \zeta^2} = 0. \quad (4.8)$$

The terms on the right-hand side of eqs (4.5)—(4.8) are given functions of zero's order quantities, and the equations of the first order are linear. The related boundary conditions for the first order are as follows:

$$\zeta = 0: U_1(\xi, 0) = 0, W_1(\xi, 0) = 0, \Theta_1(\xi, 0) = 0; \quad (4.9)$$

$$\zeta = \eta_0: W_1(\xi, \eta_0) = U_1(\xi, \eta_0) \frac{d\eta_0}{d\xi} + U_0(\xi, \eta_0) \frac{d\eta_1}{d\xi}, \quad (4.10)$$

$$\begin{aligned} \frac{\partial U_1(\xi, \eta_0)}{\partial \zeta} &= -\frac{\partial \Theta_1(\xi, \eta_0)}{\partial \xi} - \frac{\partial \Theta_1(\xi, \eta_0)}{\partial \zeta} \frac{d\eta_0}{d\xi} \\ &+ \frac{1}{2} \frac{\partial U_0(\xi, \eta_0)}{\partial \zeta} \left[1 + 4\beta_2 \frac{\partial U_0}{\partial \zeta} \right] \frac{d\eta_0}{d\xi} - \frac{\partial \Theta_0(\xi, \eta_0)}{\partial \zeta} \frac{d\eta_1}{d\xi}, \end{aligned} \quad (4.11)$$

$$P_1(\xi, \eta_0) = -\alpha \frac{d^2 \eta_1}{d\xi^2} + (\beta_1 + 2\beta_2) \left(\frac{\partial U(\xi, \eta_0)}{\partial \xi} \right)^2, \quad (4.12)$$

$$\frac{\partial \Theta_1(\xi, \eta_0)}{\partial \xi} = -Bi\Theta_1(\xi, \eta_0) - \frac{Bi}{2} [\Theta_0(\xi, \eta_0) - \Theta_g] \frac{d\eta_0}{d\xi}. \quad (4.13)$$

It is noted that the boundary conditions of the first order relationships are also linear.

Temperature equation (4.8) can be solved under boundary conditions (4.9) and (4.13), and we have

$$\Theta_1(\xi, \zeta) = f_1(\xi) \zeta, \quad f_1(\xi) = -\frac{Bi}{2} \frac{\Theta_0(\xi, \eta_0) - \Theta_g}{1 + Bi\eta_0} \frac{d\eta_0}{d\xi}. \quad (4.14)$$

Integration of eq. (4.7) yields

$$P_1(\xi, \zeta) = (\beta_1 + 2\beta_2) \left(\frac{\partial U_0}{\partial \xi} \right)^2 + P_*(\xi), \quad (4.15)$$

and function $P_*(\xi)$ is determined by boundary condition (4.12). Substituting (4.15) into eq. (4.6), we have the equation for velocity:

$$\frac{\partial^2 U_1}{\partial \zeta^2} = \beta_2 \left[- \left(U_0 \frac{\partial}{\partial \xi} + W_0 \frac{\partial}{\partial \zeta} \right) \frac{\partial U_0}{\partial \zeta} - 2 \frac{\partial U_0}{\partial \xi} \frac{\partial U_0}{\partial \zeta} + 2 \frac{\partial}{\partial \xi} \left(\frac{\partial U_0}{\partial \zeta} \right)^2 \right] + \frac{dP_*}{d\xi}. \quad (4.16)$$

The first term on the right-hand side of relation (4.15) gives the contribution of the second difference of normal stresses, which is small and can be omitted in most cases of non-Newtonian fluid. By using boundary condition (4.12), (4.15) gives

$$P_*(\xi) = -\alpha \frac{d^2 \eta_1}{d\xi^2}. \quad (4.17)$$

By using the zero order's relation (4.2) and (4.3), the solution of (4.16) is

$$\begin{aligned} U_1(\xi, \zeta) = & \alpha \left(\eta_0 \zeta - \frac{\zeta^2}{2} \right) \frac{d^3 \eta_1}{d\xi^3} - f_0 \zeta \frac{d\eta_1}{d\xi} - \frac{\alpha}{2} \zeta \eta_0 \frac{d\eta_0}{d\xi} \frac{d^3 \eta_0}{d\xi^3} + \frac{1}{2} A \frac{d\eta_0}{d\xi} - \zeta \frac{d}{d\xi} (f_1 \eta_0) \\ & + \beta_2 \left\{ \left[-\frac{\zeta^5}{15} \frac{df_5}{d\xi} + \frac{\zeta^4}{6} \frac{df_4}{d\xi} + \frac{\zeta^3}{6} \frac{df_3}{d\xi} + \frac{\zeta^2}{2} \frac{df_2}{d\xi} \right] \right. \\ & \left. - \left[f_6 - \frac{1}{3} \frac{df_5 \eta_0^4}{d\xi} + \frac{2}{3} \frac{df_4 \eta_0^3}{d\xi} + \frac{1}{2} \frac{df_3 \eta_0^2}{d\xi} + \frac{df_1 \eta_0}{d\xi} \right] \zeta \right\}. \quad (4.18) \end{aligned}$$

Furthermore, (4.5) gives

$$\begin{aligned} W_1(\xi, \zeta) = & \frac{\alpha}{6} (\zeta^3 - 3\zeta^2 \eta_0) \frac{d^4 \eta_1}{d\xi^4} - \frac{\zeta^2}{2} \frac{d\eta_0}{d\xi} \frac{d^3 \eta_1}{d\xi^3} + \frac{f_0 \zeta^2}{2} \frac{d^2 \eta_1}{d\xi^2} \\ & + \frac{\zeta^2}{2} \frac{df_0}{d\xi} \frac{d\eta_1}{d\xi} + \frac{\zeta^2}{2} \frac{d}{d\xi} \left[\frac{\alpha \eta_0}{2} \frac{d\eta_0}{d\xi} \frac{d^3 \eta_0}{d\xi^3} - \frac{1}{2} A \frac{d\eta_0}{d\xi} + \frac{d}{d\xi} (f_1 \eta_0) \right] \end{aligned}$$

$$\begin{aligned}
& +\beta_2 \left\{ \left[\frac{\zeta^6}{90} \frac{df_5}{d\xi} - \frac{\zeta^5}{30} \frac{df_4}{d\xi} - \frac{\zeta^4}{24} \frac{df_3}{d\xi} - \frac{\zeta^3}{6} \frac{df_2}{d\xi} \right] \right. \\
& \left. + \left[\frac{df_6}{d\xi} - \frac{1}{6} \frac{df_5 \eta_0^4}{d\xi} + \frac{1}{3} \frac{df_4 \eta_0^3}{d\xi} + \frac{1}{4} \frac{df_3 \eta_0^2}{d\xi} + \frac{1}{2} \frac{df_2 \eta_0}{d\xi} \right] \zeta^2 \right\}, \quad (4.19)
\end{aligned}$$

where the functions f_i ($i = 2, 3, 4, 5$) in solutions (4.19) and (4.20) are defined as follows:

$$\begin{aligned}
f_2(\xi) &= (4A - U_s) \frac{dA}{d\xi}, \quad f_3(\xi) = \alpha(U_s - 4A) \frac{d^4 \eta_0}{d\xi^4} - \left(3A + 4\alpha \frac{d^3 \eta_0}{d\xi^3} \right) \frac{dA}{d\xi}, \\
f_4(\xi) &= \alpha \left[2\alpha \frac{d^3 \eta_0}{d\xi^3} \frac{d^4 \eta_0}{d\xi^4} + \frac{d}{d\xi} \left(A \frac{d^4 \eta_0}{d\xi^4} \right) \right], \quad (4.20)
\end{aligned}$$

$$f_5(\xi) = \alpha^2 \frac{d^3 \eta_0}{d\xi^3} \frac{d^4 \eta_0}{d\xi^4}, \quad f_6(\xi) = 2 \left(\frac{\partial U(\xi, \eta_0)}{\partial \xi} \right)^2 \frac{d\eta_0}{d\xi}. \quad (4.21)$$

Functions $f_0(\xi)$ and $f_1(\xi)$ are given respectively in (3.12) and (4.14).

5 First order profile of film thickness

The solutions of the first order velocity and pressure depend on the first order film thickness η_1 . Substituting (4.18) and (4.19) into boundary condition (4.10), the equation for η_1 is demonstrated as follows:

$$\begin{aligned}
& \frac{d^4 \eta_1}{d\xi^4} + \frac{3}{\eta_0} \frac{d\eta_0}{d\xi} \frac{d^3 \eta_1}{d\xi^3} - \frac{3f_0}{2\alpha} \frac{d^2 \eta_1}{d\xi^2} + \frac{3}{\alpha} \left[\frac{A}{\eta_0^2} - \frac{1}{2\eta_0} \frac{df_0}{d\xi} - \frac{f_0}{\eta_0^2} \frac{d\eta_0}{d\xi} - \frac{\alpha}{2\eta_0} \frac{d^3 \eta_0}{d\xi^3} - \frac{U_s}{\eta_0^3} \right] \frac{d\eta_1}{d\xi} \\
& = \frac{3}{2\alpha \eta_0} \frac{d}{d\xi} \left[\frac{\alpha \eta_0}{2} \frac{d\eta_0}{d\xi} \frac{d^3 \eta_0}{d\xi^3} - \frac{A}{2} \frac{d\eta_0}{d\xi} + \frac{df_1 \eta_0}{d\xi} \right] - \frac{3\beta_2}{\alpha} \left[\frac{df_6}{d\xi} + f_6 \eta_0 \frac{d\eta_0}{d\xi} \right] \\
& + \frac{3}{\alpha} \left[\frac{\alpha}{2\eta_0} \left(\frac{d\eta_0}{d\xi} \right)^2 \frac{d^3 \eta_0}{d\xi^3} - \frac{1}{2\eta_0^2} \left(\frac{d\eta_0}{d\xi} \right)^2 + \frac{1}{\eta_0^2} \frac{d\eta_0}{d\xi} \frac{df_1 \eta_0}{d\xi} \right] \\
& + \frac{3\beta_2}{\alpha} \left\{ \left[-\frac{7\eta_0^3}{45} \frac{df_5}{d\xi} + \frac{3\eta_0^2}{10} \frac{df_4}{d\xi} + \frac{5\eta_0}{24} \frac{df_3}{d\xi} + \frac{1}{3} \frac{df_2}{d\xi} \right] + \left[-\frac{14}{15} f_5 \eta_0^2 + \frac{3}{2} f_4 \eta_0 + \frac{5}{6} f_3 + \frac{f_2}{\eta_0} \right] \frac{d\eta_0}{d\xi} \right\}. \quad (5.1)
\end{aligned}$$

Thickness equation (5.1) is a linear ordinary equation of the fourth order, and all the coefficients are given functions of zero's order solutions. The last two terms on the right-hand side of (5.1) give the influence of rheology. The related boundary conditions for η_1 can be given as

$$\eta_1(0) = 0, \quad \frac{d\eta_1(0)}{d\xi} = 0, \quad \frac{d^2 \eta_1(0)}{d\xi^2} = 0, \quad \frac{d^3 \eta_1(0)}{d\xi^3} = 0. \quad (5.2)$$

Eq. (5.1) is solved under boundary conditions (5.2).

A typical case of the zero's order solution is discussed. The typical parameters are adopted as

$$\alpha = 0.5, \quad \Theta_g = 0.1, \quad \Theta_\ell = 0.1, \quad U_s = 4, \quad B_i = 0.5. \quad (5.3)$$

The typical boundary conditions for zero's order problem are given as

$$\eta_0(0) = 1, \quad \frac{d\eta_0(0)}{d\xi} = 0.5, \quad \frac{d^2\eta_0(0)}{d\xi^2} = 0, \quad \eta_0(1) = 1, \quad (5.4)$$

and the temperature distribution at the solid boundary is

$$\Theta_s = 1 - (1 - \Theta_\lambda)\xi. \quad (5.5)$$

The profile of zero's order thickness is solved in the same way as in ref. [1], and the solution of case (5.4) is shown in fig.2.

By using the solutions of the zero's order relationships, the first order equation (5.1) is solved under boundary conditions (5.2), and the solutions for $\beta_2 = 0, -0.2, -0.4, -0.6$ and -0.8 are given in fig. 3. The values of height profile η_1 for the case of $\beta_2 = 0$ are nearly zero, and the height profiles for cases of non-zero β_2 give the influence of rheological fluid. The results show that the rheological effect enlarges the cross-section of the liquid jet, and the conclusion agrees with the usual one.

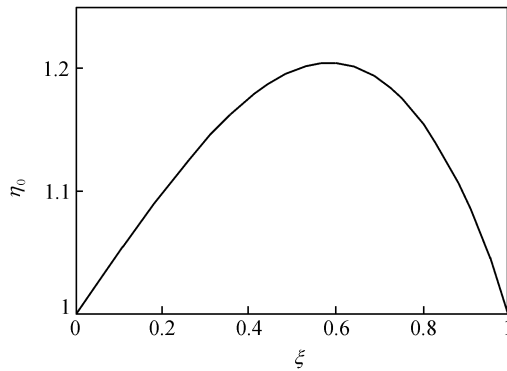


Fig. 2. Zero's order cross-section profile of liquid jet in case of $\eta_0(0) = 1, \eta_0'(0) = 0.5, \eta_0''(0) = 0$ and $\eta_0(1) = 1$.

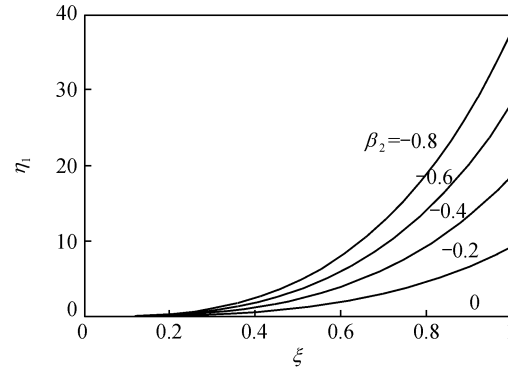


Fig. 3. First order cross-section profiles of liquid jet for $\beta_2 = 0, -0.2, -0.4, -0.6$ and -0.8 .

6 Discussions

The lubrication approximation and the perturbation method are employed to discuss the problem of jet liquid film of a non-Newtonian fluid, and the analytical solutions of temperature, pressure and velocity depending on the height of liquid film have been obtained. The results show that the influence of thermocapillary effect is to enlarge the cross-section of the jet liquid due to the heat transfer from the solid boundary to the melt, and the crossing of the free surface determines obviously the velocity distribution. Furthermore, the pressure distribution is mainly determined by the rheology property of the non-Newtonian fluid, which also enlarges the cross-section

of the liquid jet. The results of the present paper show mainly the mechanism of the cross-section variation due to both the thermocapillary effect and the rheological effect of a non-Newtonian fluid.

A non-Newtonian fluid of the second-order fluid model is assumed in the present paper, and this model has obviously limitations and cannot be applied to many sorts of polymers. However, the polymers are various, and some of the polymers can even be described by the Newtonian fluid. The discussion of the present paper only applies to special sorts of polymer that can be described approximately by the second-order fluid model. The jet process of the weak non-Newtonian approximation discussed in the present paper simplifies the complex process, and the zero's order solution reduces to the Newtonian fluid as discussed in ref. [4]. The results show that as in the case of Newtonian fluid, the heat transfer may induce the thermocapillary flow and may increase the cross-section of the liquid jet even for a non-Newtonian fluid.

It should be noted that the approximations of lubrication theory, perturbation theory and weak non-Newtonian approximations have obvious limitations. The Barus effect can have larger variation of cross-section, which may be two or three times that of the cross-section of the vessel exit, and the region of the enlarging cross-section is near the melt vessel exit. More studies should be conducted, especially in connection with the manufacture processing. The conclusions are especially for the case of a weak non-Newtonian model for simplification of the mathematical treatment, and more attention should be paid to the rheological fluid of other models related to the real cases of polymer application.

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