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An Extension of the Two-Dimensional JKR Theory to the Case with a Large Contact Width *

CHEN Shao-Hua(陈少华)**, PENG Zhi-Long(彭志龙)

LNM, Institute of Mechanics, Chinese Academy of Sciences, Beijing, 100190

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In the Hertz and JKR theories, parabolic assumptions for the rounded profiles of the sphere or cylinder are adopted under the condition that the contact radius (width) should be very small compared to the radius of the sphere or cylinder. However, a large contact radius (width) is often found in experiments even under a zero external loading. We aim at extending the plane strain JKR theory to the case with a large contact width. The relation between the external loading and the contact width is given. Solutions for the Hertz, JKR and rounded-profile cases are compared and analyzed. It is found that when the ratio of a/R is approximately larger than about 0.4, the parabolic assumptions in the Hertz and JKR theories are no longer valid and the exact rounded profile function should be used.

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Contact mechanics pioneered by Hertz[1] has been widely applied in many branches of engineering, particularly in the studies of tribology and indentation. Since the 1970s, molecular interactions between contacting objects have also been incorporated into contact mechanics models. Johnson *et al.*[2] developed the JKR model of adhesive contact based on a balance between elastic and surface energies. On the other hand, Derjaguin *et al.*[3] proposed the DMT model in which the stress field remains in the Hertz profile within the contact region while intermolecular adhesion is assessed outside the contact area. A more general model (MD model) was developed by Maugis[4] who showed that the JKR and DMT models can in fact be unified within a Dugdale type of cohesive model of adhesive contact.

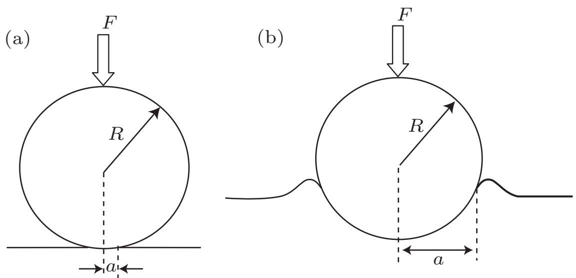

Fig. 1. Plane strain model of an elastic cylinder of radius R in adhesive contact with an elastic half-space. (a) The contact half-width a is very small and the classical two-dimensional JKR theory can be adopted. (b) The contact half-width a is much larger, which results in the classical JKR theory invalid.

The adhesive contact mechanics represented by JKR and DMT models has triggered extensive research efforts over the past three decades.[5−14] In almost all the works, parabolic approximation for the rounded profile of the sphere or cylinder is adopted, which is only valid for the cases with small contact radii as shown in Fig. 1(a). However, many experiments have found that small particles could have a large contact radius in adhesive contact with elastic substrates, even under a zero external loading[15,16] as shown schematically in Fig. 1(b). Extension of the classical JKR theory to the case with a large contact radius is needed. The sphere case has been successfully extended by Maugis.[17] In this Letter, we extend the two-dimensional plane strain JKR theory (a long cylinder in adhesive contact with a half space) to the corresponding two-dimensional case with a large contact width.

The normal displacement along the contact interface between an elastic cylinder and an elastic half-space can be written as

$$u_y = \delta_y - f(x), \tag{1}$$

where δ_y denotes the relative displacement of the centers of the cylinder and the half space during contact formation, $f(x)$ is a function describing the profile of the rounded cylinder.

The normal displacement along the contact interface can be related to the interfacial normal traction $p(x)$ via Green's functions as

$$\frac{1}{\pi E^*}\int_{-a}^{a}\frac{p(s)}{x-s}ds = \frac{\partial u_y}{\partial x}, \tag{2}$$

where the effect of tangential traction in the contact region can be neglected according to Johnson[18] and Chen and Gao.[19] E^* denotes the effective Young's modulus, which is composed of Young's moduli E_1 and E_2, and Poisson's ratios ν_1 and ν_2, of the cylinder

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**To whom correspondence should be addressed. Email: chenshaohua72@hotmail.com

and the half space, $1/E^* = (1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2$.

Combining Eqs. (1) and (2) yields

$$\frac{1}{\pi E^*}\int_{-a}^{a}\frac{p(s)}{x-s}ds = -f'(x). \tag{3}$$

In both Hertz and JKR theories, the parabolic approximation, i.e., $f(r) = r^2/(2R)$ or $f(x) = x^2/(2R)$, for the rounded profile of the sphere (3D case) or the cylinder (2D case), was used, which is only valid for the case with a small contact radius. For the case with a large contact radius, the profile should be described by an exact function $f(x) = R - \sqrt{R^2 - x^2}$ and the differential of the profile function with respect to x is

$$f'(x) = \frac{x}{\sqrt{R^2 - x^2}}. \tag{4}$$

Solving Eqs. (3) and (4) subject to the boundary condition $\int_{-a}^{a} p(x)dx = F$ yields the solution to the interfacial normal traction $p(x)$. The whole solving process is standard but very complex. Similar method has been used by Chen and Gao,[19] so that we skip all the details here and present the final interfacial tractions in the contact region,

$$p(x) = \frac{-E^*}{2\pi(a^2-x^2)^{\frac{1}{2}}}\int_{-a}^{a}\frac{s(a^2-s^2)^{\frac{1}{2}}}{(x-s)\sqrt{R^2-s^2}}ds + \frac{F}{\pi(a^2-x^2)^{\frac{1}{2}}}. \tag{5}$$

From Eq. (5), one can see that the normal traction in the contact region is singular, which is very similar to that of an interface crack model in fracture mechanics. According to the knowledge in fracture mechanics, the stress intensity factor near the contact edges can be obtained as

$$\begin{aligned} K_{\rm I} &= -\lim_{x\to a}\sqrt{2\pi(a-x)}p(x) \\ &= \frac{E^*}{2\sqrt{\pi a}}\int_{-a}^{a}\frac{s(a^2-s^2)^{1/2}}{(a-s)\sqrt{R^2-s^2}}ds - \frac{F}{\sqrt{\pi a}}, \end{aligned} \tag{6}$$

where the negative sign in front of the right side is due to the definition that compressive traction is assumed to be positive.

The dynamic Griffith energy balance criterion can be expressed as

$$G = \frac{K_{\rm I}^2}{2E^*} = w, \tag{7}$$

where w is the work of adhesion, $w = w_1 + w_2 - 2w_{12}$, w_1 and w_2 are the intrinsic surface energies of the two solids, and w_{12} is the surface energy of the contact interface.

Substituting the stress intensity factor in Eq. (6) into the Griffith energy balance criterion yields the controlling equation

$$\frac{1}{2E^*}\left[\frac{E^*}{2\sqrt{\pi a}}\int_{-a}^{a}\frac{s(a^2-s^2)^{\frac{1}{2}}}{(a-s)\sqrt{R^2-s^2}}ds - \frac{F}{\sqrt{\pi a}}\right]^2 = w, \tag{8}$$

which relates the contact half-width a to the external loading F, so that the external loading can be explicitly expressed as a function of the contact half-width a as

$$\frac{F}{E^*R} = \frac{a^2}{2R^2}\int_{-1}^{1}\frac{t(1-t^2)^{1/2}}{(1-t)\sqrt{1-a^2t^2/R^2}}dt - \sqrt{\frac{2\pi a w}{E^*R^2}}. \tag{9}$$

Let us introduce the dimensionless parameters

$$Y = \frac{F}{E^*R}, \quad X = \frac{a}{R}, \quad m = \sqrt{\frac{2\pi w}{E^*R}}. \tag{10}$$

With these notations, Eq. (9) becomes

$$Y = \frac{1}{2}X^2\int_{-1}^{1}\frac{t(1-t^2)^{1/2}}{(1-t)\sqrt{1-X^2t^2}}dt - mX^{1/2}. \tag{11}$$

For the case with a small contact radius, i.e., $a \ll R$, we have

$$\int_{-1}^{1}\frac{t(1-t^2)^{1/2}}{(1-t)\sqrt{1-a^2t^2/R^2}}dt \approx \frac{\pi}{2}. \tag{12}$$

Equation (9) will reduce to the classical plane strain JKR solution[7,9] as

$$\frac{F}{E^*R} = \frac{\pi a^2}{4R^2} - \sqrt{\frac{2\pi a w}{E^*R^2}}. \tag{13}$$

Using the dimensionless parameters in Eq. (10), Eq. (13) can be rewritten as

$$Y = \frac{\pi}{4}X^2 - mX^{1/2}. \tag{14}$$

Figure 2 shows the relation between the dimensionless external force $F/(E^*R)$ and the dimensionless contact half-width a/R for both the classical JKR and the present theories with various values of the parameter m. From the numerical calculations, one can see that the JKR approximation is valid with less than 4% relative error of $F/(E^*R)$, only when the ratio of a/R is smaller than about 0.4 for each value of parameter m. If a/R is larger than 0.4, relative errors will increase and the real rounded profile should be considered to find the correct contact solution.

Next, we concentrate on the extension of the two-dimensional Hertz solution. In the classical plane strain or three-dimensional Hertz solution, molecular interaction force between contact surfaces is not considered, i.e., the work of adhesion w is zero, which means $m = 0$. Following from Eq. (11) we have

$$Y = \frac{1}{2}X^2\int_{-1}^{1}\frac{t(1-t^2)^{1/2}}{(1-t)\sqrt{1-X^2t^2}}dt, \tag{15}$$

which describes the extension of plane strain Hertz solution.

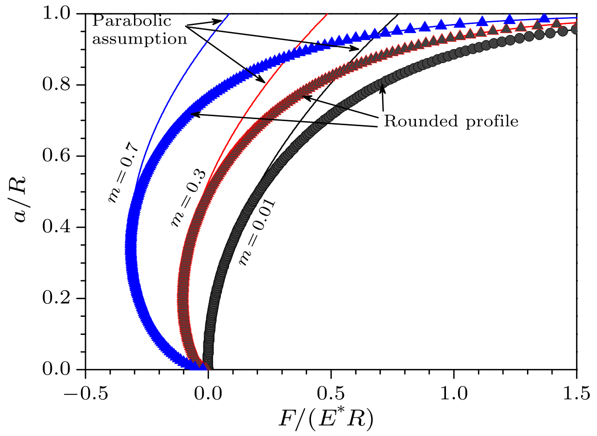

Fig. 2. Dimensionless external loading $F/(E^*R)$ as a function of the dimensionless contact half-width a/R predicted by the classical JKR and the present theories for different values of parameter m.

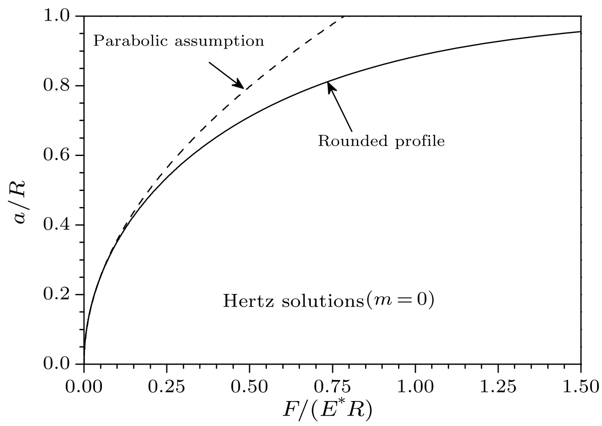

Fig. 3. Dimensionless external loading $F/(E^*R)$ as a function of the dimensionless contact half-width a/R in the form of Hertz contact predicted with the parabolic assumption and the real rounded profile function, respectively.

The classical plane strain Hertz solution can be obtained from Eq. (14) as

$$Y = \frac{\pi}{4} X^2. \tag{16}$$

Figure 3 shows the relation between the dimensionless external force $F/(E^*R)$ and the dimensionless contact half-width a/R for both the classical Hertz solution and its extension with a real rounded profile function, from which one can see that even in the Hertz solution, when the ratio of a/R is larger than about 0.4, the parabolic assumption can no longer be valid and the exact rounded profile function should be used to find the contact solution.

For the case of zero applied load, the dimensionless contact half-width a_0/R under zero load ($F = 0$) can be found for adhesive contact models with different profile functions. For the case with a parabolic assumption, the explicit solution to the dimensionless contact half-width can be expressed as

$$X = \left(\frac{16m^2}{\pi^2}\right)^{1/3}, \tag{17}$$

which is identical to the classical plane strain JKR solution,[7,9]

$$a_0 = \left(\frac{32R^2 w}{\pi E^*}\right)^{1/3}. \tag{18}$$

However, for the case with a real rounded profile, the corresponding contact half-width can be obtained from Eq. (11) and expressed by an implicit equation,

$$m = \frac{1}{2} X^{3/2} \int_{-1}^{1} \frac{t(1-t^2)^{1/2}}{(1-t)\sqrt{1-X^2t^2}} dt. \tag{19}$$

Numerical calculation is used to solve Eq. (19). The dimensionless contact half-width as a function of the parameter m is shown in Fig. 4. One can see that when $m = \pi/4$, the classical JKR solution would predict $a_0 = R$, i.e., the contact half-width equals to the cylinder radius due to the surface-energy driven.[20] In fact, the classical plane strain JKR solution is invalid theoretically without the condition of a very small contact width. While in the case with a real rounded profile function, only when $m \to \infty$, $a_0 = R$ can be asymptotically realized.

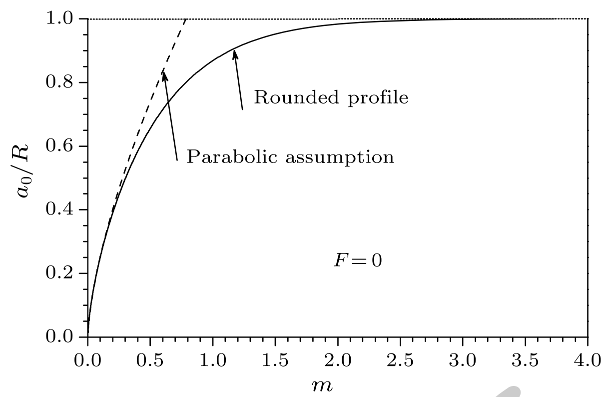

Fig. 4. Relation between the dimensionless contact half-width a_0/R and the parameter m predicted by theories with parabolic assumption and real rounded profile function, respectively, where a_0 denotes the contact half-width under zero external loading.

In conclusion, for the plane strain adhesive contact model between a cylinder and a soft elastic substrate, the ratio of the contact width to the cylinder radius can be so large that the parabolic approximation for the cylinder profile in the classical JKR theory is no longer valid. The use of an exact expression for the cylinder profile allows the classical JKR theory to be extended to the case with a larger contact width. It is found that when the ratio of a/R is approximately smaller than about 0.4, the parabolic assumption in the classical two-dimensional Hertz and JKR theories can be reasonable to approximate the rounded profile.

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