NEW TRENDS IN FLUID MECHANICS RESEARCH
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Characteristics of Flow Fields Induced by Interfacial Waves in Two-Layer Fluid

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Abstract When designing deep ocean structures, it is necessary to estimate the effects of internal waves on the platform and auxiliary parts such as tension leg, riser and mooring lines. Up to now, only a few studies are concerned with the internal wave velocity fields. By using the most representative two-layer model, we have analyzed the behavior of velocity field induced by interfacial wave in the present paper. We find that there may exist velocity shear of fluid particles in the upper and lower layers so that any structures in the ocean are subjected to shear force nearby the interface. In the meantime, the magnitude of velocity for long internal wave appears spatially uniform in the respective layer although they still decay exponentially. Finally, the temporal variation for Stokes and solitary waves are shown to be of periodical and pulse type.

Key words: internal waves, velocity field, solitary waves, two-layer, water densities

INTRODUCTION

The ocean is not always homogenous. In all water stratifications, the two-layer model represents the most intense density change. The wavelengths and wave heights of internal waves are rather larger than the dimensions of surface waves and ocean structures, which accounts for why *Morison* formula is popularly used when computing internal wave force. For free surface waves, the velocity field properties of all kinds of surface waves have been sufficiently studied. There are still few articles concerned with internal wave flow fields. For the two-layer model, except for wave amplitude H, wavelengths L, there are two depth dimensions, namely, d_1 and d_2 (d_1 and water thickness ratio $r = d_1/d_2$) for upper and lower layers. Fluid velocities are mainly related with H, L, r and water density difference ratio σ=$(\rho_2 - \rho_1)/\rho_2$.

PROBLEM FORMULATION

We consider traveling periodic waves in a two-layer fluid on a horizontal impermeable bed, which can be regarded as steady flow if the coordinate system moves at the same speed as the wave. The water is assumed incompressible and bounded by two rigid walls on the upper and lower boundaries. There is experimental evidence reported by Kao et al.[1] to support the rigid-lid approximation. Then, the origin is set on the plane of water surface, the horizontal coordinate is$x$ and the vertical coordinate is z (Figure 1).

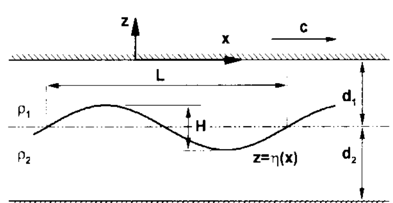

Figure 1: The coordinate of two-layer fluid interfacial waves, in the figure, the origin of the coordinate is located on the water surface, and the densities of the two layers are ρ_1 and ρ_2 respectively and the depth are d_1 and d_2

We use stream function formulation such that velocity components (u_1,v_1) and (u_2,v_2) are given by $u_1 = \partial\psi_1/\partial z$, $w_1=-\partial\psi_1/\partial x$, and $u_2 = \partial\psi_2/\partial z$, $w_2=-\partial\psi_2/\partial x$ if the motion is irrotational, ψ_1 and ψ_2 satisfy Laplace equation throughout the two fluids[2]. Thus

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial z^2} = 0 \tag{1}$$

$$\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial z^2} = 0 \tag{2}$$

The upper boundary condition is

$$\psi_1(x,0) = 0 \tag{3}$$

The bottom boundary conditions to be satisfied is

$$\psi_2(x,-d_1-d_2) = 0 \tag{4}$$

On the free interface $y = \eta(x)$, the kinematics boundary condition is

$$\psi_2\left[x,-d_1+\eta(x)\right] = -Q \tag{5}$$

in which Q equates to a positive constant, denoting the total volume rate of flow underneath the stationary wave per unit length normal to the $(x,\ z)$ plane; and the condition requiring pressure on the free surface to be constant, combined with Bernoulli's equation, that is

$$\begin{aligned}&\left[(\partial_x\psi_2)^2 + (\partial_z\psi_2)^2\right]/2 + g\left[-d_1+\eta(x)\right] - R - (1-\sigma)\left\{\left[(\partial_x\psi_1)^2 + (\partial_z\psi_1)^2\right]/2+\right.\\ &\quad\left. g\left[-d_1+\eta(x)\right] - R\right\} = 0\end{aligned} \tag{6}$$

Cheng and Li have solved the equations by perturbation expansion in terms of ε, they got the dispersion relation of two-layer fluid steady internal waves can be expressed as

$$\omega/\sqrt{gk} = c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \varepsilon^3 c_3 + \varepsilon^4 c_4 \tag{7}$$

The steam function can be gotten as

$$\begin{aligned}k\psi_1/\bar{c} &= \varepsilon f_{111}\sinh kz\cos kx + \varepsilon^2 f_{122}\sinh 2kz\cos 2kx + \varepsilon^3\left(f_{131}\sinh kz\cos kx +\right.\\ &\quad f_{133}\sinh 3kz\cos 3kx + \varepsilon^4\left(f_{142}\sinh 2kz\cos kx\cos 2kx + f_{144}\sinh 4kz\cos 4kx\right) +\\ &\quad \varepsilon^5\left(f_{151}\sinh kz\cos kx + f_{153}\sinh 3kz\cos 3kx + f_{155}\sinh 5kz\cos 5kx\right)\end{aligned} \tag{8}$$

and

$$\begin{aligned}k\psi_2/\bar{c} &= \varepsilon f_{211}\sinh k(z+h)\cos kx + \varepsilon^2 f_{222}\sinh 2k(z+h)\cos 2kx+\\ &\quad \varepsilon^3\left(f_{231}\sinh k(z+h)\cos kx+ f_{233}\sinh 3k(z+h)\cos 3kx\right) + \varepsilon^4\left(f_{242}\sinh 2k(z+h)\cos 2kx +\right.\\ &\quad f_{244}\sinh 4k(z+h)\cos 4kx) + \varepsilon^5\left(f_{251}\sinh k(z+h)\cos kx +\right.\\ &\quad f_{253}\sinh 3k(z+h)\cos 3kx + f_{255}\sinh 5k(z+h)\cos 5kx\end{aligned} \tag{9}$$

The parameters in Eq. (1~9) can refer to reference [2]. For the internal solitary waves, the governing equations and fluid velocity formulas can be found in reference [3].

INFLUENCING FACTORS

The particle velocities can be solved from Eq. (8,9). Coefficients in the eqautions are functions of kd_1d_2 and σ, which can be solved by Mathematica. Choosing phase speed c as the referential speed with kH=0.04, r=0.2, the velocities u(u_1 and u_2) can be solved as functions of z (see Figure 2),

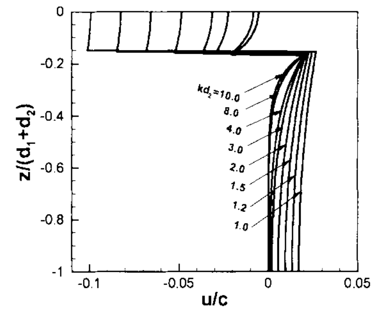

Figure 2: When $kx = 0$, the maximal fluid velocity profiles of linear periodic wave when kd_2equals to 1.0, 1.2, 1.5 etc. respectively

From Figure 2(a), we may find amplitudes of u_1 and u_2 are of opposite signs, which renders a vertical structure across the interface subject to shearing forces. For given water depth ratio r, i.e. In Figure 2, r=0.2, the upper layer is shallow water, which makes u_1 almost uniform with the increase of z; only when kd_1 (or kd_2) gets large enough, u_1 will decay a little near the pycnocline. The lower layer is deeper water, u_2 decreases rapidly with the increase of water depth, when kd_2=8.0, at the depth of $0.5d_2$, u_2 already tends to 0. Fluid vertical component w may play important role for short Stokes waves. When L is very large, the flow filed exhibits relatively uniform in the extent of $2\pi+2\pi/3<kx<2\pi+4\pi/3$.If we move z=0 to the water interface, and note the vertical coordinate as z_1, fluid velocity of linear theory can be expressed as

$$u_1 = -\varepsilon \cdot c \cdot \cos(kx - \omega t) \cosh k(z_1 - md_2) \operatorname{csc h}(mkd_2) \tag{10}$$

$$u_2 = \varepsilon \cdot c \cdot \cos(kx - \omega t) \cosh k(z_1 + d_2) \operatorname{csc h}(kd_2) \tag{11}$$

if it is deep water,

$$u_1 = -\varepsilon \cdot c \cdot e^{-kz_1} \cos(kx - \omega t) \tag{12}$$

$$u_2 = \varepsilon \cdot c \cdot e^{kz_1} \cos(kx - \omega t) \tag{13}$$

Therefore, if water depth is much larger than L, u_1 and u_2 will decrease exponentially from the interface upward and downward. Generally, water depth is much smaller than internal wavelength, which is the reason u_1 and u_2 being almost uniform in the z direction.
For the internal solitary waves, when $r > \sqrt{\rho_1/\rho_2}$, the wave is upward convex, namely polarity being positive, while $r < \sqrt{\rho_1/\rho_2}$, wave is downward concave with polarity being negative. If we let H=90m, d_1=70m, d_2=290m, the solitary wave width is W=2 000m. The computed phase speed can be got as c=1.917m/s. If the time wave passing one width distance is noted as $T = W/c$=1043.1s. The fluid velocity profiles at different time are given as the following Figure 3.

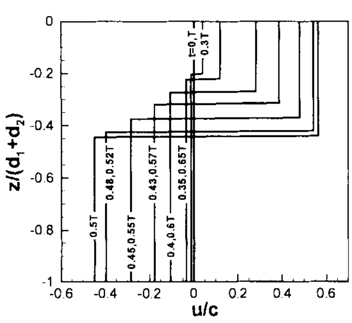

Figure 3: The maximal fluid velocity profiles of internal solitary wave when $t$equals to different values

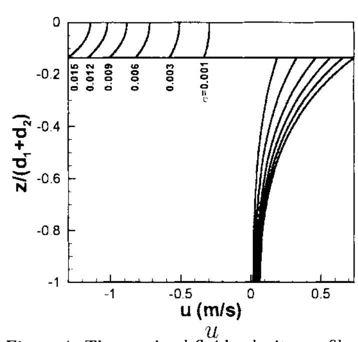

Figure 4: The maximal fluid velocity profiles of second order periodic wave when σ equals to 0.001, 0.003, 0.006, 0.009, 0.012, and 0.015 respectively

We may find on the two sides of the pycnocline, the orientations of u_1 and u_2 are always opposite to each other and almost keep consistent in its layer for any given phase angle. Fluid vertical component w is much smaller the u for the long internal waves, e.g., u_1/w_1=1.118coth(0.009x/[1.0-4.0(0.25+0.005z)]) when d_1=50, d_2=200, L=6000 and σ=0.003. It's easy to know that $|w_1|$ $|u_1|$ except when x=0.0.
In Fig. 3, $d_1/d_2 < 1.0$, so that the wave polarity is negative. We may notice that for any given phase angle, u_1 is always larger than u_2, and the orientation of phase speed c is the same with particle horizontal component. During the whole time of passing one wave width, u_1 holds its orientation, and $u_1 > u_2$ keeps. When wave polarity changes, the orientation of u_1 will be inverse to c, and $u_1 < u_2$ will hold. In the real ocean environment, the negative solitary waves are in the majority and the water depth ratio is usually smaller than 1.0.
Water density difference ratio σ in ocean circumstance is usually less than 1.0%. According to linear periodic interfacial wave dispersion relation, when σ increases, the wave phase speed c will increase too. For given water and wave conditions, the particle velocities (u) will increase too. In Figure 3, let $kd_2 = \pi$, r=0.2, and L=5 000m, we will notice that with the increase of σ, fluid velocity components u_1, $u_2$increase evidently even when σ vary from 0.001 to 0.003. As a result of phase speed change, the wave period T will decrease accordingly by showing a larger wave frequency.
Besides water depth ratio r, water densities, there are more factors such as harmonic of higher order waves, wavelength L, wave height H, the selected wave theory and phase angle θ are important to affect the structure of wave velocity fields. According to the dispersion relation, for given water and wave condition, the larger L and H, the stronger the velocity field will be. For the effects of water depth ratio m and wave polarity, it may be not right that the higher order theory will induce stronger velocity field than lower order theory.

CONCLUSIONS

For any given phase angle, the velocities in the two layers are always of opposite signs, that render the structure across the pycnocline subject to a shear force. Although, u_1 and u_2 decrease exponentially from the interface upward and downward, they seems to be uniform in relatively shallow layers. Furthermore, the polarity of interfacial waves change with depth ratio$r$ (depending on wheather it is larger or smaller than critical value r_c). Finally, Stokes and solitary waves display periodic or pulse type behavior.

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