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ELASTIC-PLASTIC CONSTITUTIVE RELATION OF PARTICLE REINFORCED COMPOSITES*

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ABSTRACT: In this paper, a systematic approach is proposed to obtain the macroscopic elastic-plastic constitutive relation of particle reinforced composites (PRC). The strain energy density of PRC is analyzed based on the cell model, and the analytical formula for the macro-constitutive relation of PRC is obtained. The strength effects of volume fraction of the particle and the strain hardening exponent of matrix material on the macro-constitutive relation are investigated, the relation curve of strain versus stress of PRC is calculated in detail. The present results are consistent with the results given in the existing references.

KEY WORDS: cell model, elastic-plastic, constitutive relation

1 INTRODUCTION

Composites comprised of a metal or intermetallic matrix reinforced by particles which do not deform plastically have properties which make them potentially attractive for a range of applications, including high specific stiffness materials and creep resistant high temperature materials, which are widely used in the production of major components in various industries, such as those of automobiles and aircrafts. The overall mechanical properties of composites are determined by mechanical properties of the matrix and the volume, size, and spacings of the reinforcement. The optimization and improvement of strength, ductility and fracture toughness of composites is a major research orientation, and the investigation on the fundamental relationships between the macroscopic behavior of the composites and its microstructure is very important.

The linear elastic theory of composites is well developed, and methods based on Eshelby's solution of a single inclusion embedded in an infinite matrix have been developed and employed by many investigators (for example, the self-consistent method, Mori-Tanaka method, and the differential method)[1~3]. In this work we focus our attention on nonlinear elastic-plastic materials and derive the overall macroscopic response of the composites through a volume averaging procedure as discussed below.

Investigation on the elastic-plastic behavior of PRC was pioneered by a number of investigators, and some analytical models were suggested. Budiansky et al.(1982)[4] (see also Duva & Hutchinson(1984)[5]) had studied the plastic deformation of an infinite solid

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containing a void. In that approach an upper bound solution was obtained where approximate velocity fields were assumed by the superposition of a uniform field due to a uniform remote axisymmetric stress in the absence of the void, and an additional nonuniform one. Then a Rayleigh-Ritz method based on minimum energy principle was used to generate approximate solutions in the material. Duva(1984)[6] used a similar method to analyze the stiffening effect of rigid inclusions on an infinite domain of a power-law material. Since interactions between neighboring inclusions were ignored, the validity of the equation was limited to small concentrations of inclusion.

Zhu & Zbib(1995)[7] investigated the mechanical properties of composites based on the use of a finite unit cell model accounting for the interaction of particles in composites with periodic microstructures(e.g. Bao et al. (1991)[8] and Llorca & González(1998)[9]), the finite unit cell model is geometrically clearer and can describe the microstructure features more accurately.

This work is motivated by the work of Gurson(1977)[10] on yield surfaces for porous materials. It seems that, although many aspects of PRC are well explained, understanding of the strengthening mechanisms is still a subject that needs further investigation. The purpose of the present work is to develop a mathematical model capable of incorporating the basic features of elastic-plastic properties of PRC. The main goals are to rigorously derive the relationships that describe the overall nonlinear elastic-plastic macro constitutive behavior of PRC, and propose explicit expressions for the macro-constitutive relation of the composites, accounting for various materials parameters.

2 CELL MODEL OF PRC

In general, an arbitrary and non-uniform distribution of particles, which may include local clustering, may better represent the actual state of the composite. The size and geometry of the particles are inhomogeneous which will significantly affect the mechanical properties of composites, especially the strength, ductility and fracture toughness. However if all the actual factors were considered, the problem will become very difficult. In this paper, the composites were idealized as uniformly distributed periodic arrays of unit cells, and each unit cell consists of an elastic inclusion surrounded by an elastic-plastically deforming matrix (see Fig.1).

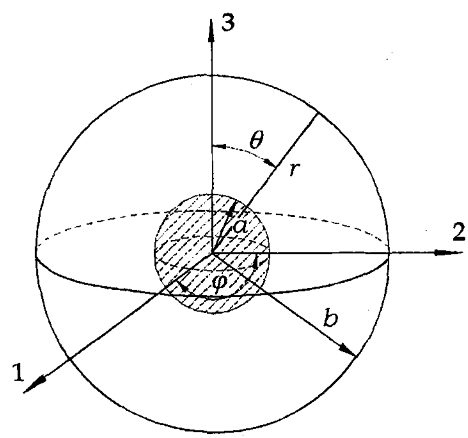

Fig.1 The cell model for particle reinforcement composite

We suppose that the macro elastic-plastic constitutive relation can be expressed with the global theory as follows

$$E_{ij} = E_{ij}^e + E_{ij}^p = C_{ijkl}\Sigma_{kl} + \frac{3}{2}\frac{E_e^p}{A(E_e^p)}S_{ij} \tag{1}$$

where Σ_{ij} is the macro average stress of the composite, S_{ij} is the deviatoric part of the macro average stress, $E_{ij}, E^e_{ij}, E^p_{ij}$ are the macro strain, elastic strain and plastic strain, respectively. $E^p_e = \left(\frac{2}{3}E^p_{ij}E^p_{ij}\right)^{1/2}$ is the macro equivalent plastic strain, and C_{ijkl} is the macro elastic compliance tensor of the composite. According to Mori-Tanaka equation (Mori & Tanaka (1973)[11]), we have

$$\boldsymbol{L} = \boldsymbol{L}_0(\boldsymbol{I} + f\boldsymbol{Q})^{-1} \tag{2}$$

where $\boldsymbol{L}$ is the macro elastic modulus matrix of composite, i.e., the inverse matrix of the macro elastic compliance matrix, $\boldsymbol{L}_0$ is the elastic modulus matrix of matrix material, $\boldsymbol{I}$ is the unit matrix, f is the volume fraction of particle reinforcement and $\boldsymbol{Q}$ is the concentration factor matrix, which can be expressed as

$$\boldsymbol{Q} = \{\boldsymbol{L}_0 + (\boldsymbol{L}_1 - \boldsymbol{L}_0)[f\boldsymbol{I} + (1-f)\boldsymbol{S}]\}^{-1}(\boldsymbol{L}_0 - \boldsymbol{L}_1) \tag{3}$$

where $\boldsymbol{L}_1$ is elastic modulus matrix of the particle, $\boldsymbol{S}$ is Eshelby's tensor matrix.

In Eq.(1), the function $A(E^p_e)$ describes the work hardening properties of composites. It is worth noting that the constitutive relation of the composites will be completely determined if $A(E^p_e)$ is known.

Considering the cell model, as shown in Fig.1, a spherical inclusion embedded in the spherical matrix, and the inclusion is well cohered with the matrix. Duva and Hutchinson (1984)[5] have shown that the macro constitutive relation of composites can be expressed by the macro strain energy density if the constitutive relation of any components can be expressed by the strain energy density, which is expressed as

$$W = \sum_{k=0,1}^{n} \frac{1}{V}\int_{V_k} w_k \mathrm{d}V \tag{4}$$

where V_0 is the volume of matrix in the unit cell, w_0 is the micro strain energy density of matrix, $V_k(k \neq 0)$ is the volume of the kth particle component, w_k is the micro strain energy density of the kth particle component. The macro constitutive relation can be expressed as

$$\Sigma_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{5}$$

where the macro strain of composites, E_{ij}, is

$$E_{ij} = \frac{1}{V}\int_V \varepsilon_{ij}\mathrm{d}V \tag{6}$$

and the macro stress of composites, Σ_{ij}, is

$$\Sigma_{ij} = \frac{1}{V}\int_V \sigma_{ij}\mathrm{d}V \tag{7}$$

In this paper, the elastic deformations of both inclusion and matrix were neglected to simplify the evaluation of the strain hardening function of composites, then the particle can be simplified as a rigid inclusion, and the matrix was simplified as an incompressible power law material, which obeys the following equation

$$\frac{\sigma}{\sigma_0} = \left(\frac{\varepsilon}{\varepsilon_0}\right)^n \tag{8}$$

where ε_0 is the reference strain, σ_0 is the reference stress, and n is the strain hardening exponent of the matrix material. For multi-axis stress condition, Eq.(8) can be written as

$$s_{ij} = \frac{2}{3}\frac{\sigma_0}{\varepsilon_e}\left(\frac{\varepsilon_e}{\varepsilon_0}\right)^n \varepsilon_{ij} \tag{9}$$

where s_{ij}, ε_{ij} is the deviatoric part of microscopic stress and strain, respectively, and $\varepsilon_e = \left(\frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}\right)^{1/2}$ is the effective strain, so the strain energy density of the matrix is

$$w_0 = \int s_{ij}(\varepsilon)\mathrm{d}\varepsilon_{ij} = \frac{\sigma_0\varepsilon_0}{n+1}\left(\frac{\varepsilon_e}{\varepsilon_0}\right)^{n+1} \tag{10}$$

since $w_k = 0$ $(k \neq 0)$, according to Eq.(4) the macroscopic strain energy density is

$$W = \frac{1}{V}\int_{V_0} w_0 \mathrm{d}V \tag{11}$$

and then

$$W = \frac{1}{V}\left(\frac{\sigma_0\varepsilon_0^{-n}}{n+1}\right)\int_{V_0} \varepsilon_e^{n+1}\mathrm{d}V \tag{12}$$

since the composite was treated as incompressible plastic material, then the macroscopic stress is

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{13}$$

3 ANALYSES METHOD AND RESULTS

Suppose that the outer boundary condition of the unit cell is

$$u_i = E_{ij}X_j \qquad X \subset \Omega_1 \tag{14}$$

and since the particle was assumed as a rigid inclusion, so the inner boundary condition of cell is

$$u_i = 0 \qquad X \subset \Omega_2 \tag{15}$$

where Ω_1 and Ω_2 are the outer and inner boundaries of the unit cell, respectively.

The strain field equations of unit cell written in spherical coordinate are

$$\left.\begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r} & \varepsilon_\theta &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\ \varepsilon_\varphi &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\theta}{r}\mathrm{ctg}\theta + \frac{u_r}{r} & \gamma_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{aligned}\right\} \tag{16}$$

where

$$u_\varphi = 0 \qquad \gamma_{r\varphi} = \gamma_{\theta\varphi} = 0 \tag{17}$$

For plastically incompressible matrix materials under axisymmetric loading conditions, a simple way to derive the physical components of the displacement vector $\boldsymbol{u}$ is to employ a displacement potential function ζ such that $\boldsymbol{u} = \nabla \times (0, 0, \zeta/\sqrt{g_{33}})$(see, for example, Lee & Mear (1992)[12]), yielding

$$u_r = \sqrt{\frac{g_{11}}{g}}\frac{\partial \zeta}{\partial \theta} \qquad u_\theta = -\sqrt{\frac{g_{22}}{g}}\frac{\partial \zeta}{\partial r} \qquad u_\varphi = 0 \tag{18}$$

where g_{11}, g_{22}, g_{33} are the covariant components and g is the determinant of the metric tensor of the spherical coordinate systems. Consider the choice of the displacement potential function $\zeta(r,\theta)$, a general form for a spherical model was given by Budiansky et al. (1982)[4] as

$$\zeta(r,\theta) = A_0 \mathrm{ctg}\theta + \sum_{k=2,4,\ldots} \mathrm{P}_{k,\theta}(\cos\theta) f_k(r) \tag{19}$$

where P_k is the Legendre polynomial of degree k, f_k is a function of radius r alone, and A_0 corresponds to the volume deformation. In this work, the following form of ζ is adopted

$$\zeta(r,\theta) = \frac{\sqrt{g}}{6r^2} \sum_{k=2,4,\ldots} \sum_{m=-\infty}^{\infty} \overline{E} r^3 \Big(1 - \frac{a}{r}\Big)^2 \Big(\frac{a}{r}\Big)^m \beta_{km} \sin k\theta \tag{20}$$

As we know in spherical coordinate, $\sqrt{g} = r^2 \sin\theta$, $\sqrt{g_{11}} = 1$ and $\sqrt{g_{22}} = r$, then according to Eq.(18) and Eq.(20), the displacements of cell are

$$\left.\begin{aligned} u_r &= \sum_{k=2,4,\ldots} \sum_{m=-\infty}^{\infty} \frac{1}{6}\overline{E} r \Big(1 - \frac{a}{r}\Big)^2 \Big(\frac{a}{r}\Big)^m \beta_{km} (\mathrm{ctg}\theta \sin k\theta + k \cos k\theta) \\ u_\theta &= -\sum_{k=2,4,\ldots} \sum_{m=-\infty}^{\infty} \frac{1}{6}\overline{E} r \Big[(m-1)\frac{a}{r} - (m-3)\Big] \Big(1 - \frac{a}{r}\Big) \Big(\frac{a}{r}\Big)^m \beta_{km} \sin k\theta \\ u_\varphi &= 0 \end{aligned}\right\} \tag{21}$$

Equation (21) shows that the inner boundary condition (15) is automatically satisfied. Expanding Eq.(14) in spherical coordinates and noting that the macroscopic strain E_{ij} has only principal values $E_{11} = E_{22}$ and E_{33}, we obtain the outer boundary conditions

$$\left.\begin{aligned} u_r &= \frac{\overline{E}}{6} b (1 + 3\cos 2\theta) \\ u_\theta &= -\frac{\overline{E}}{2} b \sin 2\theta \end{aligned}\right\} \qquad r = b \tag{22}$$

where

$$\overline{E} = E_{33} - \frac{1}{2}(E_{11} + E_{22}) \tag{23}$$

Upon applying the boundary conditions (22) to Eq.(21) we obtain

$$\left.\begin{aligned} &\Big(1 - \frac{a}{b}\Big)^2 \sum_{m=-\infty}^{\infty} \Big(\frac{a}{b}\Big)^m \beta_{km} = 1 && k = 2 \\ &\sum_{m=-\infty}^{\infty} \Big(\frac{a}{b}\Big)^m \beta_{km} = 0 && k = 4, 6, \ldots \\ &\Big(1 - \frac{a}{b}\Big) \sum_{m=-\infty}^{\infty} \Big[(m-1)\frac{a}{b} - (m-3)\Big] \Big(\frac{a}{b}\Big)^m \beta_{km} = 3 && k = 2 \\ &\sum_{m=-\infty}^{\infty} \Big[(m-1)\frac{a}{b} - (m-3)\Big] \Big(\frac{a}{b}\Big)^m \beta_{km} = 0 && k = 4, 6, \ldots \end{aligned}\right\} \tag{24}$$

So the strain field of the unit cell can be evaluated with the following equations,

$$\left.\begin{aligned}
\varepsilon_r &= \sum_{k=2,4,\ldots}\sum_{m=-\infty}^{\infty} -\frac{1}{6r^2}\overline{E}(a-r)\left(\frac{a}{r}\right)^m (a+am+r-mr)(\mathrm{ctg}\theta\sin k\theta + k\cos\theta)\beta_{km} \\
\varepsilon_\theta &= \sum_{k=2,4,\ldots}\sum_{m=-\infty}^{\infty} \frac{1}{6r^2}\overline{E}(a-r)\left(\frac{a}{r}\right)^m (k(am-(-2+m)r)\cos k\theta + \\
&\quad (a-r)\mathrm{ctg}\theta\sin k\theta)\beta_{km} \\
\varepsilon_\varphi &= \sum_{k=2,4,\ldots}\sum_{m=-\infty}^{\infty} \frac{1}{6r^2}\overline{E}(a-r)\left(\frac{a}{r}\right)^m (k(a-r)\cos k\theta + (am- \\
&\quad (-2+m)r)\mathrm{ctg}\theta\sin k\theta)\beta_{km} \\
\gamma_{r\theta} &= -\sum_{k=2,4,\ldots}\sum_{m=-\infty}^{\infty} \frac{1}{6r^2}\overline{E}\left(\frac{a}{r}\right)^m (-k(a-r)^2\cos k\theta\mathrm{ctg}\theta + (a^2(-1+m+ \\
&\quad m^2+k^2) - 2a(-1+(-1+m)m+k^2)r + (1+(-3+m)m+k^2)r^2 + \\
&\quad (a-r)^2\mathrm{ctg}^2\theta)\sin k\theta)\beta_{km}
\end{aligned}\right\} \tag{25}$$

Let

$$\varepsilon_{ij} = \overline{E}\hat{\varepsilon}_{ij} \tag{26}$$

then Eq.(12) is turned into

$$W = \frac{1}{V}\left(\frac{\sigma_0\varepsilon_0^{-n}}{n+1}\right)\overline{E}^{n+1}\int_{V_0}\hat{\varepsilon}_e^{n+1}\mathrm{d}V \tag{27}$$

Substituting Eq.(27) into Eq.(13), we have

$$S_{ij} = \frac{1}{V}\frac{\sigma_0\overline{E}^n}{\varepsilon_0^n}\int_{V_0}\hat{\varepsilon}_e^{n+1}\mathrm{d}V\frac{\partial\overline{E}}{\partial E_{ij}} \tag{28}$$

then

$$S_{11} = S_{22} = -\frac{1}{2}S_{33} = -\frac{1}{2V}\frac{\sigma_0\overline{E}^n}{\varepsilon_0^n}\int_{V_0}\hat{\varepsilon}_e^{n+1}\mathrm{d}V \tag{29}$$

Since $E_e = 2\overline{E}/3$, $\Sigma_e = S_{33} - S_{11}$, so

$$\Sigma_e = \left(\frac{3}{2}\right)^{n+1}\frac{1}{V}\frac{\sigma_0 E_e^n}{\varepsilon_0^n}\int_{V_0}\hat{\varepsilon}_e^{n+1}\mathrm{d}V \tag{30}$$

and after simplifying, we have

$$\frac{\Sigma_e}{\sigma_0} = F(f,n)\left(\frac{E_e}{\varepsilon_0}\right)^n \tag{31}$$

and

$$F(f,n) = \left(\frac{3}{2}\right)^{n+1}\frac{1}{V}\int_{V_0}\hat{\varepsilon}_e^{n+1}\mathrm{d}V \tag{32}$$

where $f = a^3/b^3$. According to Eq.(31), the work hardening function is obtained as

$$A(E_e) = \sigma_0 F(f,n)\left(\frac{E_e}{\varepsilon_0}\right)^n \tag{33}$$

Here, we call $F(f,n)$ the hardening factor. It is a nonlinear problem to solve for $F(f,n)$. The minimum energy principle was employed in evaluating the unknown coefficients of displacement field of unit cell. The central issue of the problem is to evaluate the integral

$$\hat{W} = \frac{1}{V}\int_{V_0} \hat{\varepsilon}_e^{n+1} \mathrm{d}V \tag{34}$$

where the integral must be carried out numerically. The minimization of $\hat{W}$ is a minimum problem with the boundary conditions, where Eq.(24) were rewritten as $\sum b_i(\beta_{km})$, then the problem is transferred to the following equivalent minimum problem

$$\Psi(\Delta\beta_{km}, \lambda_i) = \hat{W}(\Delta\beta_{km}) + \sum \lambda_i b_i(\Delta\beta_{km}) \tag{35}$$

where λ_i is Lagrangian factor. Newton-Raphson method is the common method used to solve this kind of problem, but the calculating work is rather heavy, and the convergence of the solution depends on the initial value of solution. If the initial value is not selected properly, the right solution could not be reached. An adaptive method is adopted in this paper recommended as follows. Firstly, a displacement mode consisting of two terms was adopted, i.e., in Eq.(21), $k = 2$, and $m = 1, 2$, where the two coefficients β_{21} and β_{22} can be determined by boundary conditions, then the strain of any point of the cell was obtained, and then the integral of Eq.(35) can be worked out easily. Secondly, based on the solution of two terms of displacement mode, the coefficients of multi-terms displacement mode can be obtained by using the perturbation method. The steps are as follows

Suppose that

$$\hat{\varepsilon}_{ij} = \hat{\varepsilon}_{ij*} + \Delta\hat{\varepsilon}_{ij} \tag{36}$$

where the symbol '*' express the solution related to the displacement field of two terms, then

$$\hat{\varepsilon}_{ij}^2 = \hat{\varepsilon}_{ij*}^2 + 2\hat{\varepsilon}_{ij*}\Delta\hat{\varepsilon}_{ij} + (\Delta\hat{\varepsilon}_{ij})^2 \tag{37}$$

then

$$\hat{\varepsilon}_e^{n+1} = \hat{\varepsilon}_{e*}^{n+1}\left(1 + \frac{4}{3}\frac{\hat{\varepsilon}_{ij*}\Delta\hat{\varepsilon}_{ij}}{\hat{\varepsilon}_{e*}^2} + \frac{2}{3}\frac{\Delta\hat{\varepsilon}_{ij}\Delta\hat{\varepsilon}_{ij}}{\hat{\varepsilon}_{e*}^2}\right)^{(n+1)/2} \tag{38}$$

Expanding Eq.(38) using binomial expansion, with the first two terms of the expansion being reserved, we obtain

$$\hat{\varepsilon}_e^{n+1} \cong \hat{\varepsilon}_{e*}^{n+1}\left[1 + \frac{2(n+1)}{3}\frac{\hat{\varepsilon}_{ij*}\Delta\hat{\varepsilon}_{ij}}{\hat{\varepsilon}_{e*}^2} + \frac{n+1}{3}\frac{\Delta\hat{\varepsilon}_{ij}\Delta\hat{\varepsilon}_{ij}}{\hat{\varepsilon}_{e*}^2} + \frac{n^2-1}{8}\left(\frac{4}{3}\frac{\hat{\varepsilon}_{ij*}\Delta\hat{\varepsilon}_{ij}}{\hat{\varepsilon}_{e*}^2}\right)^2\right] \tag{39}$$

Substituting Eq.(39) into Eq.(35), and differentiating Ψ with respect to β_{km} and λ_i the linear algebraic equations about β_{km} will be found by letting the derivative of Ψ to be zero. β_{km} can be obtained by solving the linear equations.

The number of terms of the displacement field included in the calculation should be considered seriously, which may affect the computing precision and calculation work. If the number of terms of the displacement field is not large enough, the precision of calculation could not reach the required level, however, if the number of terms of displacement is very large, the calculation work will become difficult.

According to the work of Budiansky et al.(1982)[4], high order harmonics have miner quantitative influence on the solution which involves volume averaging. Duva(1984)[6] used the first two order harmonics about θ, and Zhu & Zbib(1995)[7] selected the first order of the harmonic, they all obtained satisfied results. The results of the first order harmonic included in the calculation is compared with the results of the first two order harmonics included, the error involved is less than 0.1%, see Fig.2, so in the calculation of this paper, we use the first order harmonic, i.e., $k = 2$.

Theoretically, we need to consider a large number of terms in the series (21) to ensure a good quantitative accuracy. The terms with positive and nonpositive powers should both be considered, i.e., a finite number of terms in the series (21) with $M_1 \le m \le M_2 (M_1 \le 0, M_2 \ge 1)$. To ensure the required precision, the effect of the number of terms in m on the precision of calculation is also investigated by comparing the results of selecting $m = -3, \ldots, 7$ with those of selecting $m = -3, \cdots, 8$. The calculating results were described in Fig.3. The difference between them was less than 0.1%, so we choose $m = -3, \cdots, 8$ (i.e., $M_1 = -3, M_2 = 8$) in the following calculation.

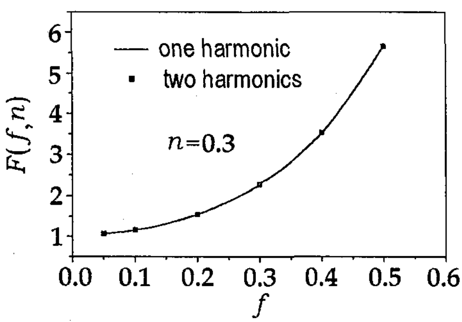

Fig.2 The influence of harmonics included in the displacement field on the calculating precision of $F(f, n)$, $m = -1, \cdots, 4$

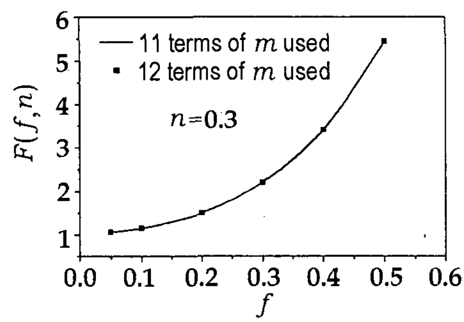

Fig.3 The dependence of solutions on the number of terms of displacement field respect m

The hardening factor $F(f, n)$ was evaluated at a various value of particle volume fraction f , in this case the parameter of the matrix was given by $n = 0.1$. The predicted relation of $F(f, n)$ versus f was consistent with the results given by Zhu & Zbib (1995)[7], and the comparing result was plotted in Fig.4. It can be seen that the particles have a distinct hardening effect on the composites as its volume fraction increases.

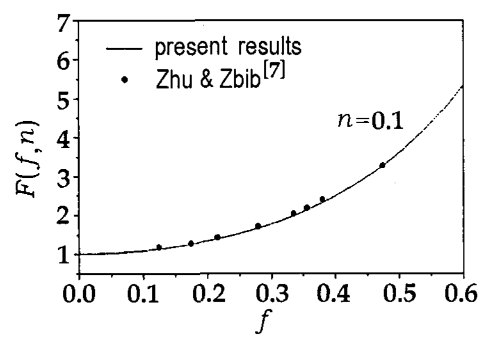

Fig.4 Hardening effect of particle volume fraction on the strength of composite

The relation curve of stress and plastic strain of composites Al-SiC was calculated for two different particle volume fractions, namely, 0.19 and 0.31. The materials constants were selected from reference of Zhu & Zbib(1995)[7], the stress and plastic strain relation of matrix is expressed as, $\sigma = \sigma_0 \varepsilon^n$, where $\sigma_0 = 0.1226\,\text{GPa}$ and $n = 0.09835$. The present results agree well with the results given by Zhu & Zbib as shown in Fig.5.

The elastic-plastic relation of strain and stress of composites was investigated. For the composite Al-Al_2O_3 , the elastic constants of the matrix, Young's modulus and Poisson's ratio, were 70 GPa and 0.33, while those of the reinforcement were 450 GPa and 0.17. The flow stress for the matrix was represented by the power law $\sigma = \sigma_0 \varepsilon^n$, with $n = 0.2$, $\sigma_0 = 700\,\text{GPa}$. Two different volume fractions of particle were considered, i.e., $f = 0.05$ and 0.15, the calculating results were described in Fig.6 and Fig.7, respectively. The calculation results were compared with the finite element results of Llorca & González(1998)[9]. From the two figures, it can be seen that the analytical results were well supported by the FEM results when the particle volume fraction is small, however, some error exists when the particle volume fraction is large. The discrepancy can be attributed mainly to three factors, the first one is that the cell model of this paper is different from that of Llorca & González(1998)[9] which is a spherical inclusion embedded in the cylindrical matrix, the second one is that the strain formula used in FEM calculation is selected as logarithm strain which is different from that of this paper, the third one is that the macroscopic stress of the reference is the average stress on the transverse cross section of cell, which is also different from this paper.

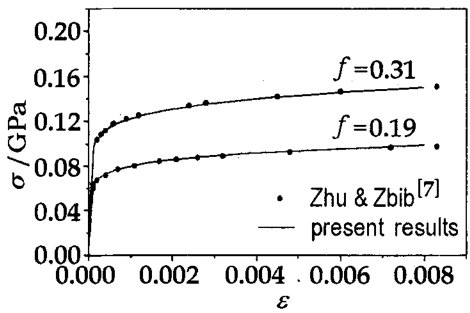

Fig.5 Predicted stress and plastic strain curves compared to the calculation results of Zhu & Zbib[7]

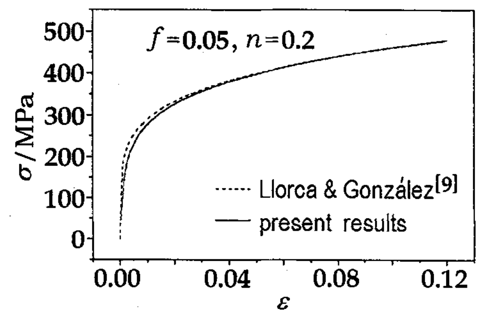

Fig.6 Predicted stress and elastic-plastic strain curve compared with FEM results of Llorca & González[9], ($f = 0.05$)

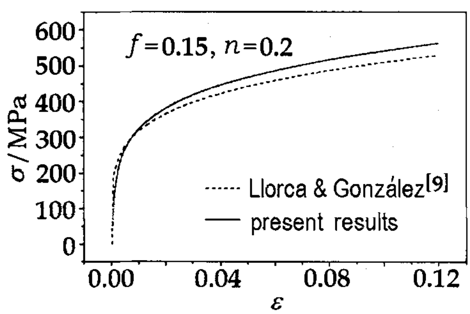

Fig.7 Predicted stress and elastic-plastic strain curve compared with FEM results of Llorca & González[9], ($f = 0.15$)

4 ELASTIC-PLASTIC CONSTITUTIVE RELATION

The relation of hardening factor $F(f,n)$ versus particle volume fraction f was calculated at a various value of strain hardening exponent n, the calculating results were described in Fig.8, which shows that the influence of strain hardening exponent and the particle volume fraction on the macroscopic constitutive relation is significant. Numerical calculation was not the goal of this paper, the objective of this paper is to give the analytical expression of macroscopic constitutive relation of PRC. So the calculation results plotted in Fig.8 were fitted by the polynomial formulae, then the analytical formula of $F(f,n)$ was obtained as follows

$$F(f,n) = A_0 + A_1 f + A_2 f^2 + A_3 f^3 + A_4 f^4 \tag{40}$$

where

$$\left.\begin{aligned}
A_0 &= 1.000\,14 - 0.000\,41n + 0.001\,54n^2 - \\
&\quad 0.002\,54n^3 + 0.001\,32n^4 \\
A_1 &= 0.505\,67 + 1.850\,7n - 0.441\,87n^2 + \\
&\quad 1.138\,36n^3 - 0.566\,39n^4 \\
A_2 &= 1.390\,65 + 2.243\,55n + 28.400\,18n^2 - \\
&\quad 35.719\,9n^3 + 19.178\,01n^4 \\
A_3 &= 3.109\,18 + 45.158\,05n - 182.950\,22n^2 + \\
&\quad 322.200\,01n^3 - 173.039\,26n^4 \\
A_4 &= 31.773\,42 - 47.341\,4n + 517.894\,51n^2 - \\
&\quad 798.393\,58n^3 + 459.754\,51n^4
\end{aligned}\right\} \tag{41}$$

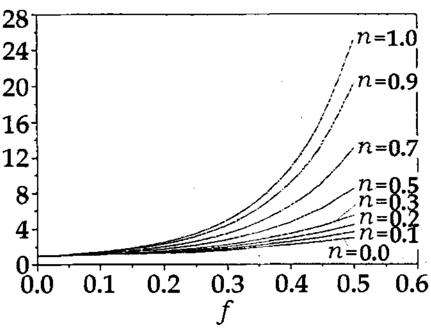

Fig.8 Effects of strain hardening exponent and particle volume fraction on the constitutive relation of composite

This fitting results in Eq.(40) and Eq.(41) provide an approximation to $F(f,n)$ for PRC in the range $f \le 0.2$ and $0 \le n \le 1.0$. The explicit expression of macroscopic elastic-plastic constitutive relation of PRC was obtained by combining Eqs.(1), (33) and (40).

To check the precision of the fitting formulae, the hardening factor $F(f,n)$ was re-calculated by using the analytical fitting equations at $n = 0.1, 0.3, 0.5, 1.0$, the results were compared with the exact value plotted in the Fig.9, which shows a good fitting to the exact value. Zhu & Zbib (1995)[7] also gave an analytical formula of $F(f,n)$, however it is only used in the range $f \le 0.2$ and $0 \le n \le 0.2$, where the value range of n is narrower than that of this paper. In the intersection of the fitting range of the two kinds of analytical formulae, the two fitting equations are consistent with each other as shown in Fig.10.

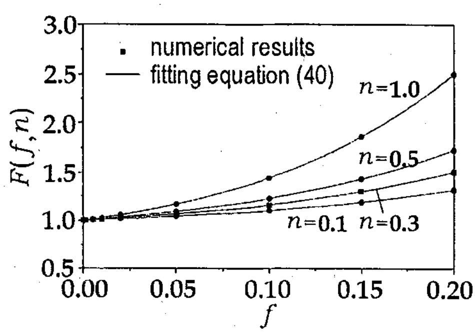

Fig.9 Predicted results of analytical formula of $F(f,n)$ compared with the exact values

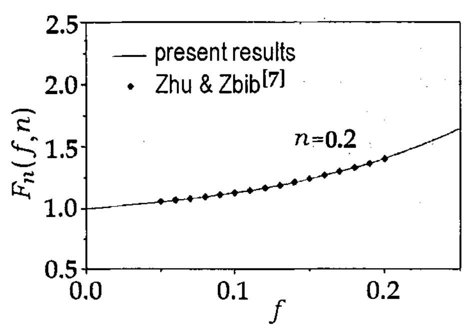

Fig.10 Predicted results of present analytical formula compared with the results of formula of Zhu & Zbib[7]

5 CONCLUSION

In summary, this paper found the macroscopic elastic-plastic constitutive relation of PRC based on the cell model, and gave the solving method, i.e., first suppose the displacement field with two terms and deduce the strain and stress field, then the displacement field with multi-terms was obtained by perturbation method based on the two terms results, and then hardening factor $F(f, n)$ was evaluated, without solving the nonlinear problem. The analytical formula of constitutive relation of PRC was given further in a wider range of usage. It is shown that this model has successfully described the basic features of the nonlinear elastic-plastic response of the composite and identified effects of particle shape and volume fraction on the overall flow properties of composite. The results predicted by the current work are consistent with the results given in the references of Zhu & Zbib (1995)[7], and Llorca & González (1998)[9], which proved that current model is right.

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